ARithmetische Geometrie OberSeminar

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Categorical Local Langlands

The local Langlands conjectures aim at an understanding of the representation theory of p-adic groups. More precisely, for a reductive group G over $\mathbb{Q}_p$ – following our main sources, assumed to be split in this seminar – one is interested in the representation theory of the totally disconnected group $G(\mathbb{Q}_p)$ on $\mathbb{C}$-vector spaces; these are required to be "smooth" in the sense that any vector has an open stabilizer in $G(\mathbb{Q}_p)$. Traditionally, the local Langlands conjectures aim at a classification of the irreducible smooth representations of $G(\mathbb{Q}_p)$ (in terms of L-parameters). More recently, various motivations have led to the desire to describe instead the whole category of smooth representations of $G(\mathbb{Q}_p)$. Again, the hope is that this category admits a "Langlands dual" description in terms of certain data on the side of L-parameters; and that the classification of irreducibles is a consequence.

Precise conjectures in this spirit have been proposed, by Hellmann [7], Zhu [9], and Fargues–Scholze [5]; and some of Hellmann's conjectures have been proved by Ben-Zvi–Chen–Helm–Nadler [2]. Our goal in this seminar is to understand Hellmann's formulation of these conjectures, and the results of Ben-Zvi–Chen–Helm–Nadler.

Recall that an L-parameter for G is (a $\hat{G}(\mathbb{C})$-conjugacy class of) a pair (ϕ, N) where

$$\phi : W_{\mathbb{Q}_p} \to \hat{G}(\mathbb{C})$$

is a continuous map of groups from the Weil group $W_{\mathbb{Q}_p}$ into the Langlands dual group $\hat{G}$ of G, and $N \in \mathrm{Lie}\hat{G}$ is a nilpotent element such that for all $w \in W_{\mathbb{Q}_p}$, one has $\mathrm{Ad}(\phi(w))(N) = |w|N$ where $|\cdot| : W_{\mathbb{Q}_p} \to p^{\mathbb{Z}}$ is the norm map. With this definition, L-parameters naturally live in a moduli space $\mathbb{L}_{\hat{G}}$, an Artin stack over $\mathbb{C}$, which is an infinite disjoint union of complete intersection affine schemes of dimension $\hat{G}$, modulo conjugation by $\hat{G}$.

Conjecture 1 ([7]). *There is a fully faithful functor*

$$D^+(\mathrm{Rep}_{\mathbb{C}} G(\mathbb{Q}_p)) \to D^+_{\mathrm{qcoh}}(\mathbb{L}_{\hat{G}}),$$

satisfying various properties; in particular, a normalization condition for tori, and compatibility with parabolic induction.

After a choice of Whittaker datum for G, this functor should in fact be canonical.

The conjecture says that, at least after passage to derived categories, the theory of smooth representations of $G(\mathbb{Q}_p)$ embeds into the theory of (quasi)coherent sheaves on the stack $\mathbb{L}_{\hat{G}}$.[1][2]

For $G = \mathrm{GL}_n$, and to some extent for other groups, it is possible to reduce this conjecture to its version for the principal block. Recall that Bernstein decomposed $\mathrm{Rep}_{\mathbb{C}} G(\mathbb{Q}_p)$ into a direct product of so-called Bernstein blocks. In particular, there is the block containing the trivial representation, known as the principal block. It can equivalently be described

[1] The conjectures in [9] and [5] enlarge the left-hand side so as to conjecturally realize an equivalence.

[2] At this point, it is not clear how to compare the natural t-structures under this equivalence, and for this reason the conjecture, as it stands, does not imply a description of the irreducible smooth representations.

as the subcategory of $\mathrm{Rep}_{\mathbb{C}} G(\mathbb{Q}_p)$ of those representations that are generated by their fixed vectors under an Iwahori subgroup $I \subset G(\mathbb{Q}_p)$; and this category is equivalent to representations of the Iwahori–Hecke algebra $\mathcal{H}(G,I)$. The Iwahori–Hecke algebra admits an explicit description in terms of generators and relations, and has been the subject of intense study in representation theory.

On the side of L-parameters, one can similarly restrict attention to those parameters (ϕ, N) where ϕ is trivial on the inertia subgroup of $W_{\mathbb{Q}_p}$; in this case, ϕ is equivalently given by an element of $\hat{G}$ (the image of Frobenius). The corresponding part of $\mathbb{L}_{\hat{G}}$ is thus given by the space $\mathbb{L}_{\hat{G}}^{\mathrm{unip}}$ of pairs

$$(g,N) \in \hat{G} \times \mathrm{Lie}\hat{G}, \mathrm{Ad}(g)(N) = pN,$$

up to $\hat{G}$-conjugation; these are also known as unipotent Langlands parameters. Again, this is very explicit. In this situation, the conjecture has been proved:

Theorem 2 ([2]). *There is a fully faithful functor*

$$D^+(\mathcal{H}(G,I)) \to D^+_{\mathrm{qcoh}}(\mathbb{L}_{\hat{G}}^{\mathrm{unip}})$$

compatible with parabolic induction.

In fact, such a functor is essentially determined by the image $\mathcal{H}(G,I)$, and compatibility with parabolic induction pins down what the image has to be; it is a certain canonical object known as the coherent Springer sheaf. Then the theorem essentially comes down to showing that the (derived) endomorphisms of the Springer sheaf are given by $\mathcal{H}(G,I)$.

Remark 3. *An interesting open conjecture is that the coherent Springer sheaf is actually a sheaf, i.e. concentrated in cohomological degree* 0.

To define the coherent Springer sheaf, one uses the map $\mathbb{L}_{\hat{B}} \to \mathbb{L}_{\hat{G}}$ where $\hat{B} \subset \hat{G}$ is a Borel. Unfortunately, it turns out that $\mathbb{L}_{\hat{B}}$ has to be considered as a derived stack in general; thus, some amount of derived algebraic geometry is required.

Ben-Zvi–Chen–Helm–Nadler prove the conjecture using some further techniques from derived algebraic geometry, in particular the use of Hochschild homology, or its refinement, of categorical traces. Another ingredient is a theorem of Bezrukavnikov describing the category of Iwahori-equivariant sheaves on the affine flag variety.

Talks

If you like to give a talk, please indicate your preferences by filling out the form `https://forms.gle/s9JbsQdCPaQGcdBY9` by Wednesday, October 12.

Talk 1: Smooth representation theory, principal block

Recall basics on the smooth representation theory of p-adic groups, and then describe the principal block. In particular, show that the category of representations generated by their Iwahori fixed vectors is stable under passage to subquotients, and equivalent to the category of representations of the Iwahori–Hecke algebra. Describe this algebra by generators and relations. Discuss the example of GL_2 in detail, and describe all irreducible representations that have Iwahori fixed vectors.

Talk 2: L-parameters

Describe the space of L-parameters, and show that it is reduced and a complete intersection of the expected dimension. Concentrate on the space of unipotent parameters.

Discuss some examples, especially for GL_2. Also discuss the related spaces for parabolic subgroups, and show by examples that in this case their dimension can be too big. Reference: [7, Section 2.1 until 2.11]; reducedness in general comes from [1], see [4, Proposition 2.7].

Talk 3: Coherent Springer sheaf

Define the coherent Springer sheaf. In particular, define the structure of derived scheme (or stack) on the space of parameters for parabolic subgroups. Discuss the conjecture that it is concentrated in degree 0, and the example of GL_2 (and GL_3 if time permits). Reference: [7, Section 2, starting with 2.12].

Talk 4: The conjecture for GL_n, especially $n = 2$

Following [7, Section 3.1, Conjecture 3.2], state the categorical conjecture for the unipotent block. In the case of GL_n, explain the conjectural relation to the modified local Langlands correspondence, and to the Emerton–Helm family, [7, Conjecture 4.8, Conjecture 4.14, Remark 4.17]. For $n = 2$, prove all conjectures, [7, Section 4.7]. Moreover, discuss in detail how various objects are matched under the equivalence; in particular, for $n = 2$, describe the image of all irreducible smooth representations.

Talk 5: The Kazhdan–Lusztig isomorphism

Explain the isomorphism between the Grothendieck group of $\hat{G} \times \mathbb{G}_m$-equivariant coherent sheaves on the Steinberg variety $\mathcal{Z}$ (for the dual group), and the Hecke algebra (with q as formal variable), as stated in [2, Theorem 1.2]; see [8, Theorem 3.5].

Talk 6: The big lines of the proof in [2]

Explain the general strategy of the proof in [2]. Roughly, one considers the monoidal DG-category of $\hat{G}$-equivariant coherent sheaves on the Steinberg variety, and takes the trace of the Frobenius endofunctor. Discuss the ambient symmetric monoidal $(\infty, 2)$-category of DG categories in which this computation takes place [6]. (For us, DG category is synonymous with $\mathbb{C}$-linear presentable stable ∞-category.)

Explain that, using Bezrukavnikov's equivalence discussed in the next talk, the proof consists of two parts: (a) Show that the categorical trace of Frobenius on Bezrukavnikov's category gives the Iwahori–Hecke algebra; and (b) that the categorical trace of Frobenius on Bezrukavnikov's category is given by the (derived) endomorphisms of the coherent Springer sheaf.

Here, (a) is dealt with using the geometric interpretation, and (b) using the spectral interpretation.

Talk 7: Bezrukavnikov's equivalence

Bezrukavnikov [3] obtained a categorification of the Kazhdan–Lusztig isomorphism; explain the statement (see [2, Theorem 1.13]). If possible, explain some parts of the proof, and be precise about how much structure is known to be compatible under the equivalence. (Is it an equivalence of ∞-categories or just triangulated categories? Is it compatible with the monoidal structure? Is it compatible with the Frobenius endomorphisms?)

Talk 8: The categorical trace of Frobenius

The goal of this talk is to prove [2, Corollary 1.17], identifying the categorical trace of Frobenius on Bezrukavnikov's category with the Iwahori–Hecke algebra. Either find a direct way to do this, or follow [2, Section 2, until Corollary 2.32] and deduce this from

Kazhdan–Lusztig's isomorphism and a comparison between K-theory and Hochschild homology.

Talk 9: Monoidal traces

For monoidal DG categories, define the notion of monoidal traces; the outcome of this is still a DG category. Compare the monoidal trace and the categorical trace, as discussed in [2, Section 1.5.2, Section 3.1] and the references cited therein.

Explain that the goal thus becomes twofold: Prove that the monoidal trace of multiplication by q on the Steinberg stack gives the derived category of coherent sheaves on the space of unipotent parameters; and identify its canonical object with the coherent Springer sheaf. This is [2, Theorem 1.19].

Talk 10: Perfect versus Coherent Complexes

For an Artin stack X (in characteristic 0) recall the DG categories $\mathrm{QC}(X) = \mathrm{Ind}(\mathrm{Perf}(X))$ and $\mathrm{QC}^!(X) = \mathrm{Ind}(\mathrm{Coh}(X))$ and recall their functorialities. Describe the effect of categorical traces in both settings, i.e. prove [2, Equation (3.1)]. Moreover, prove the "Calabi–Yau" property [2, Lemma 3.12].

These computations make fixed point spaces appear. Show that the space of unipotent parameters appears as the fixed points of multiplication by q on $\hat{\mathfrak{g}}/\hat{G}$, [2, Proposition 4.3].

Talk 11: End of proof

Prove [2, Theorem 3.25], upgrading the results of the previous talk to a similar assertion about monoidal traces, and use it to finish the proof of the main theorem, see [2, Theorem 4.12 (1)].

If time permits, say something about how to identify (anti-)spherical modules under the equivalence, and the compatibility with parabolic induction [2, Theorem 4.12 (2), (3)].

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