$_{\text{kleine}}$AG GAGA

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Introduction

At the heart of algebraic geometry lies the study of solutions of polynomial equations. If we work over the complex numbers and consider

$$X = \{x \in \mathbb{A}^n \mid f_1(x) = \cdots = f_r(x) = 0\}$$

we can also interpret the polynomials f_i as holomorphic functions on $\mathbb{C}^n$ and their zeroset as a complex space X^{an}. If the differential of $f = (f_1, \ldots, f_r)$ has everywhere maximal rank along X^{an} then, by the implicit function theorem, X^{an} is a complex manifold and can thus be studied with the methods of differential geometry.

This local picture can easily be globalised to give a functor

$$(\text{schemes of finite type}/\mathbb{C}) \longrightarrow (\text{complex analytic spaces}), \quad X \mapsto X^{an}.$$

Clearly the Zariski topology on X is much coarser than the euclidean topology on X^{an} and their local structure is completely different so the following natural questions arise:

(1) What is the relation between the local and global properties of X and those of X^{an}?
(2) Which complex spaces (or manifolds) are algebraic, that is, of the form X^{an} for some scheme X?
(3) When does a theorem proved for X resp. X^{an} translate to a theorem in the other category?

One instance of the last question is to understand for example the relation between the cohomology groups $H^i(X, \mathcal{O}_X)$ and $H^i(X^{an}, \mathcal{O}_{X^{an}})$. In general they might be different, but it turns out that for X proper resp. X^{an} compact there is a general comparison theorem for all coherent sheaves.

The first talks will be used to recall the neccessary definition and then prove the main comparison results, originally due to Serre [Ser56]. The third talk focuses on examples of manifolds that fail to be projective or algebraic in one way or the other. The last talk will show that when we go beyond the reach of the comparison theorem non-trivial differences appear. In examples we will see that there are results for projective varieties that can be proved (at least up to now) only via algebraic or analytic methods.

Talks

1. The analytic space associated to a scheme. (60 Minutes)
Recall the definition of a complex analytic space and of a coherent sheaf in that context. Mention Oka's theorem of the coherence of the structure sheaf.

The talk covers roughly the first three sections of [SGA1, XII]. Define the analytic space associated to a scheme X locally of finite type over $\mathbb{C}$ as a complex analyic space X^{an} representing the functor

$$(\text{analytic spaces})^{op} \longrightarrow (\text{sets}), \quad T \mapsto Hom_{\text{ringed spaces}/\mathbb{C}}(T, X)$$

and show that it always exists. The representing space X^{an} comes with an universal morphism of ringed spaces $\phi : X^{an} \to X$. Moreover, the induced morphism of formal completions $\hat{\phi}_x : \hat{\mathcal{O}}_{X,x} \longrightarrow \hat{\mathcal{O}}_{X^{an},x}$ is an isomorphism. Conclude that ϕ is flat.[1]

We get a morphism between categories of modules

$$\phi^* : \mathcal{O}_X\text{-}Mod \longrightarrow \mathcal{O}_{X^{an}}\text{-}Mod, \quad F \mapsto F^{an} = \phi^{-1}(F) \otimes_{\mathcal{O}_X} \mathcal{O}_{X^{an}}$$

which is faithful, exact, and transforms coherent sheaves into coherent sheaves. To avoid extensive use of EGA the speaker might want to consult [Ser56, Prop. 2.10].

In the last part of this first talk we will compare properties of schemes and morphisms with their analytic counterparts. This is summarized nicely in sections 2-3 of [SGA1, XII]. For example we have the following.

- A scheme X is regular, reduced, of dimension n if and only if X^{an} is so.
- A morphism $f : X \to Y$ is injective, surjective, proper, flat, smooth, separated, normal if and only if f^{an} is so.

As the proofs are all quite similar, we suggest to prepare a slide with all the statements and present the proofs of only one or two of them to examplify the methods.

2. GAGA theorem and coherent cohomology. (60 Minutes)

In this section we proof the main GAGA theorem.

Theorem (Serre, [Ser56]) — If X is a proper scheme over $\mathbb{C}$ then

$$\phi^* : Coh(X) \longrightarrow Coh(X^{an})$$

is an equivalence of categories.[2]

We will give a proof of this theorem in the case X is projective following Serre's original paper [Ser56][3] sections 2.12-2.17. The general case of a proper X is reduced to the projective one by using Chow's lemma (see [SGA1, XII,4.1]) and shall not be explained. The presentation of Serre is phantastic - there is nothing we can add here to make it more readable. We hope that the clear structure will enable the speaker to single out the most important steps to be presented at the blackboard in the given time.

The proof is structured into three parts. The first step is a comparison theorem for cohomology of coherent sheaves which is also of independent interest.

Theorem 1 — For all $F \in Coh(X)$ and $q \in \mathbb{Z}$ we have

$$H^q(X, F) \cong H^q(X^{an}, F^{an}).$$

This shows in particular $\Gamma(X, F) = \Gamma(X^{an}, F^{an})$. There is also a relative version [SGA1, 4.2].

The next step is fully faithfulness.

Theorem 2 — For all $F, G \in Coh(X)$ we have $Hom(F, G) \cong Hom(F^{an}, G^{an})$.

And eventually, and most difficult, the essential surjecticity.

Theorem 3 — For every coherent analytic sheaf $\mathcal{M} \in Coh(X^{an})$ there is a coherent algebraic sheaf $F \in Coh(X)$ such that $F^{an} \cong \mathcal{M}$.

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[1] Serre invented flatness in this context.

[2] A general philosophy behind this theorem is that algebraic functions or sections can be characterized among the holomorphic ones by some growth condition at infinity. E.g. $\mathbb{C}[z] \subset \mathcal{O}(\mathbb{C})$ are the holomorphic functions $f : \mathbb{C} \to \mathbb{C}$ which satisfy

$$\exists C, D > 0, n \in \mathbb{N} \text{ such that } |f(z)| \leq C||z||^n + D.$$

In the proper case, there is no "infinity" and hence every holomorphic section is algebraic. In this vein Deligne proves GAGA theorems for non-proper schemes in [Del70, II.2.22,II.2.24].

[3] or its translation.

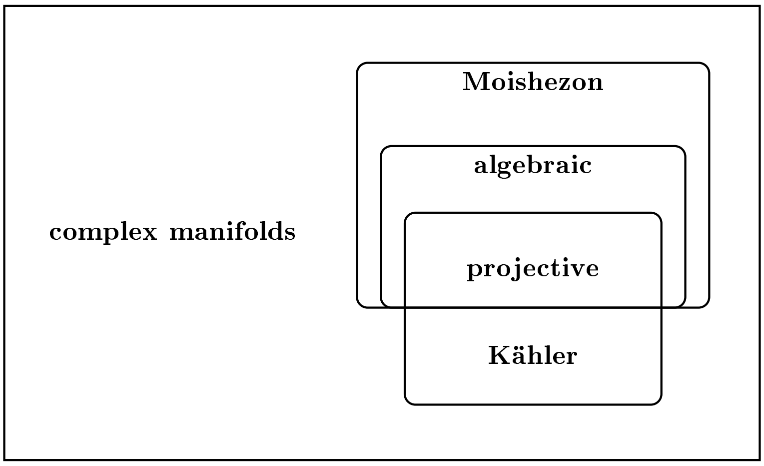

FIGURE 1. Relations between categories of complex manifolds.

3. Which manifolds are algebraic? (60 Minutes)
In this talk we look at the image of the functor (Sch/$\mathbb{C}$) $\to$ (complex spaces) in the smooth case, that is at the question which manifolds are algebraic. The relations between various categories of complex manifolds is illustrated in Figure 1.

The GAGA theorem shows in particular that *every projective complex manifold is algebraic* which was proved earlier by Chow.

Define the algebraic dimension $a(X) = tr\text{-}deg(K(X))$ where $K(X)$ is the field of meromorphic functions. Cite the result $a(X) \leq \dim X$ [4]. If equality holds we call X a Moishezon variety [Uen75, p. 24 ff.]. Prove that *algebraic implies Moishezon* and explain how to construct a Moishezon variety from a complex space via *algebraic reduction*.

Then present the two examples from [Har77, Appendix B, 3.4.1, 3.4.2] to show that *there are algebraic varieties that are not projective* and *there are Moishezon varieties that are not algebraic* . If you want, mention that Moishezon varieties are the same as Artin's algebraic spaces [Art70].

Then give your favorite definition of Kähler manifold and sketch a proof that *every projective complex manifold is Kähler*. Cite Moishezon's theorem [Moĭ66][5] that *projective is equivalent to Kähler + Moishezon*.

To construct a *Kähler manifold that is not Moishezon* we do the following:
Lemma — If X is a compact Kähler manifold with vanishing Neron-Severi group then $a(X) = 0$.

Sketch of proof: Assume there is a non-trivial meromorphic function and let D be the divisor of zeros or poles. Then $c_1(\mathcal{O}_X(D)) = [D]$ in $NS(X) = H^2(X,\mathbb{Z}) \cap H^{1,1}(X)$. Integrating a suitable power of the Kähler form over D is non-trivial, thus $[D]$ is non-trivial – a contradiction.

A concrete example is obtained by takeing X to be a general complex torus. (Reason: write $X = V/\Gamma$. Then for Γ general the intersection of $H^2(X,\mathbb{Z}) = \Lambda^2\Gamma^*$ with $H^{1,1}(X) = V \otimes \bar{V}$ in $\Lambda^2 V^* \otimes \mathbb{C} = H^2(X,\mathbb{C})$ where we identified $V \otimes \mathbb{C} = V \oplus \bar{V}$)

As an *example of a manifold that is neither Moishezon nor Kähler*, construct a Hopf surface $S = \mathbb{C}^2 \setminus \{0\}/\mathbb{Z}$ and show that $H^2(X,\mathbb{R}) = 0$ (see e.g. [BHPV, p, 225]). Conclude that S cannot be Kähler. Choose your favorite way to prove that

[4]I don't know how to call this. It is attributed to Siegel in [Huy05] while it is called Thimm's theorem in [Rem56].

[5]If your russian is not fluent see [Moi67, Chapter 1, Theorem 11].

S is not Moishezon. Suggestions: 1) Recall that smooth Moishezon surfaces are projective [BHPV, p. 161]. 2) Show that S does not contain enough curves to be birational to an algebraic surface.

If you have time left give an example of a *normal algebraic surface that is not projective*. Several are constructed in [Sch99], the classic by Hironaka is described in [BHPV, p. 161].

4. Differences and interaction. (45 Minutes)
GAGA does not take care of everything, there are some cases where the algebraic and analytic category behave differently. As an example we compare the cohomology of the sheaf of rational functions on an algebraic manifold with the cohomology of the sheaf of meromorphic functions on the associated complex manifold. Explain the relation to divisors and line bundles and present the main results of [CKL10]. Pay special attention to the fact that the exponential sequence is a tool we can only exploit in the analytic category.

If you have enough time, also give a hint the following: There are non-isomorphic algebraic varieties that become isomorphic in the analytic category [Har70, p. 232].

Conclude by mentioning the following 2 theorems that are of obvious interest but so far can only be proved in one of the categories.

Theorem (Mori '82) — Let X be a projective manifold and $C \subset X$ an irreducible curve such that $K_X.C = \deg K_X|_C < 0$. Then X contains a rational curve.

You can find a slightly more precise version in [KM98, Theorem 1.13]. The strategy of the proof is also known as *bend & break* and uses reduction modulo p.

Theorem (Siu '98) — Let $p : X \to \Delta$ be a smooth projective family of compact complex manifolds parametrized by the open unit 1-disk Δ. Assume that the fibres $X_t = p^{-1}(t)$ are of general type. Then for every positive integer m the plurigenus $\dim H^0(X_t, K_{X_t}^{\otimes m})$ is independent of $t \in \Delta$, where K_{X_t} is the canonical line bundle of X_t.

You can find this in [Siu98].

References

[Art70] M. Artin. Algebraization of formal moduli. II. Existence of modifications. *Ann. of Math. (2)*, 91:88–135, 1970.

[BHPV] Wolf P. Barth, Klaus Hulek, Chris A. M. Peters, and Antonius Van de Ven. *Compact complex surfaces*, volume 4 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 2004.

[CKL10] Xi Chen, Matt Kerr, and James D. Lewis. The sheaf of nonvanishing meromorphic functions in the projective algebraic case is not acyclic. . *C. R., Math., Acad. Sci. Paris*, 348(5-6):291–293, 2010. available at http://www.maths.dur.ac.uk/~dma0mk/MerR.pdf.

[Del70] P. Deligne *Equations Differentielles a Points Singuliers Reguliers*. Lecture Notes in Mathematics, 163, Springer Verlag 1970

[SGA1] A. Grothendieck and M. Raynaud *Revertements Etales et Groupe Fondamental (SGA1)*. Lecture Notes in Mathematics, 224, Springer Verlag, 1971, Also available on Arxiv.

[Har70] Robin Hartshorne. *Ample subvarieties of algebraic varieties*. Notes written in collaboration with C. Musili. Lecture Notes in Mathematics, Vol. 156. Springer-Verlag, Berlin, 1970.

[Har77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[Huy05] Daniel Huybrechts. *Complex geometry*. Universitext. Springer-Verlag, Berlin, 2005.

[KM98] János Kollár and Shigefumi Mori. *Birational geometry of algebraic varieties*, volume 134 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original.

[Moĭ66] B. G. Moĭšezon. On n-dimensional compact complex manifolds having n algebraically independent meromorphic functions. I. *Izv. Akad. Nauk SSSR Ser. Mat.*, 30:133–174, 1966.

[Moi67] B.G. Moishezon. On n-dimensional compact complex varieties with n algebraically independent meromorphic functions. i-iii. *Am. Math. Soc., Transl., II. Ser.*, 63:51–177, 1967.

[Rem56] Reinhold Remmert. Meromorphe Funktionen in kompakten komplexen Räumen. *Math. Ann.*, 132:277–288, 1956.

[Sch99] Stefan Schröer. On non-projective normal surfaces. *Manuscripta Math.*, 100(3):317–321, 1999.

[Ser56] J.-P. Serre *Geometire algebrique et geometrie analytique* Annales de l'institut Fourier, tome 6 (1956), Translation available at http://www.math.mcmaster.ca/ arnoldt/Serre-GAGA.dvi

[Siu98] Yum-Tong Siu, *Invariance of Plurigenera*, Inventiones Mathematicae, vol. 134, no.3 (1998).

[Uen75] Kenji Ueno. *Classification theory of algebraic varieties and compact complex spaces.* Lecture Notes in Mathematics, Vol. 439. Springer-Verlag, Berlin, 1975. Notes written in collaboration with P. Cherenack.