## Rational Cohomology of Configuration Spaces of Surfaces

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1. Introduction. The k-th configuration space $C^k(M)$ of a manifold M is the space of all unordered k-tuples of distinct points in M. In previous work [BCT] we have determined the rank of $H_*(C^k(M);\mathbb{F})$ for various fields $\mathbb{F}$. However, for even dimensional M the method worked for $\mathbb{F}=\mathbb{F}_2$ only. The following is a report on calculations of $H^*(C^k(M);\mathbb{Q})$ for M a deleted, orientable surface. This case is of considerable interest because of its applications to mapping class groups, see [BCP]. Similar results for (m-1)-connected, deleted 2m-manifolds will appear in [BCM].

2. Statement of results. The symmetric group $\Sigma_k$ acts freely on the space $\tilde{C}^k(M)$ of all ordered k-tuples $(z_1,\ldots,z_k)$, $z_i \in M$, such that $z_i \neq z_j$ for $i \neq j$. The orbit space is $C^k(M)$. As in [BCT] we will determine the rational vector space $H^*(C^k(M);\mathbb{Q})$ as part of the cohomology of a much larger space. Namely, if X is any space with basepoint $x_o$, we consider the space

$$(1) \qquad C(M;X) = \left( \coprod_{k\geq 1} \tilde{C}^k(M) \underset{\Sigma_k}{\times} X^k \right) \Big/ \approx$$

where $(z_1,\ldots,z_k;x_1,\ldots,x_k) \approx (z_1,\ldots,z_{n-1};x_1,\ldots,x_{k-1})$ if $x_k = x_o$. The space C is filtered by subspaces

$$(2) \qquad F_k C(M;X) = \left( \coprod_{j=1}^{k} \tilde{C}^j(M) \underset{\Sigma_j}{\times} X^j \right) \Big/ \approx$$

and the quotients $F_kC/F_{k-1}C$ are denoted by $D_k(M;X)$.

Let $\bar{M}_g$ denote a closed, orientable surface of genus g, and $M_g$ is $\bar{M}_g$ minus a point. We study $C(M_g;S^{2n})$ for $n\geq 1$. $H^*$ will always stand for rational cohomology, and P[ ] resp. E[ ] for polynomial resp. exterior algebras over $\mathbb{Q}$.

*Theorem A. There is an isomorphism of vector spaces*

$$(3) \qquad H^*C(M_g;S^{2n}) \cong P[v,u_1,\ldots,u_{2g}] \otimes H_*(E[w,z_1,\ldots,z_{2g}],d)$$

*with* $|v|=2n$, $|u_i|=4n+2$, $|w|=4n+1$, $|z_i|=2n+1$, *and the differential* $d$ *is given by* $d(w) = 2(z_1z_2 + \ldots + z_{2g-1}z_{2g})$.

Giving the generators weights, $\mathrm{wght}(v) = \mathrm{wght}(z_i) = 1$ and $\mathrm{wght}(u_i) = \mathrm{wght}(w) = 2$, makes $H^*C$ into a filtered vector space. We denote this weight filtration by $F_kH^*C$. The length filtration $F_kC$ of $C$ defines a second filtration $H^*F_kC$ of $H^*C$.

*Theorem B. As vector spaces*

$$(4) \qquad H^*F_kC(M_g;S^{2n}) = F_kH^*C(M_g;S^{2n}).$$

It follows that $H^*D_k(M_g;S^{2n})$ is isomorphic to the vector subspace of $H^*(g,n) = P[v,u_i] \otimes H_*(E[w,z_i],d)$ spanned by all monomials of weight exactly $k$. To obtain the cohomology of $C^k(M_g)$ itself, we consider the vector bundle

$$(5) \qquad \eta^k: \tilde{C}^k(M_g) \times_{\Sigma_k} \mathbb{R}^k \to C^k(M_g)_+$$

which has the following properties. First, the Thom space of m times $\eta^k$ is homomorphic to $D_k(M_g;S^m)$. Secondly, it has finite even order, see [CCKN]. Hence

$$(6) \qquad D_k(M_g;S^{2nk}) = \Sigma^{2n_k \cdot k}\, C^k(M_g)_+$$

for $2n_k = \mathrm{ord}(\eta^k)$. Thus we have

Theorem C. *As a vector space,* $H^*C^k(M_g)$ *is isomorphic to the vector subspace generated by all monomials of weight* $k$ *in* $H^*(g,n_k)$, *desuspended* $2n_k k$ *times.*

Regarding the homology of $E = E[w, z_1, \ldots, z_{ig}]$ we have

Theorem D. *The homology* $H_*(E,d)$ *is as follows:*

(7) $\quad \operatorname{rank} H_{i(2n+1)} = \binom{2g}{i} - \binom{2g}{i-2}$ for $i = 0,1,\ldots g$, *and all (non-zero) elements have weight* $i$;

(8) $\quad \operatorname{rank} H_{i(2n+1)+4n+1} = \binom{2g}{i} - \binom{2g}{i+2}$ *for* $i = g,\ldots,2g$, *and all (non-zero) elements have weight* $i+2$;

(9) $\quad \operatorname{rank} H_j = 0$ *in all other degrees* $j$.

Note the apparent duality $\operatorname{rank} H_j = \operatorname{rank} H_{N-j}$ for $N = 2g(2n+1)+4n+1$.

We will give the proof of Theorem A in the next section. The proof of Theorem B is the same as for [BCT, Thm.B]. By what we said above Theorem C folows from Theorem B. And Theorem D will be derived in the last section.

3. *Mapping spaces and fibrations.* Let D denote an embedded disc in $M_g$. There is a commutative diagram

$$(10) \qquad \begin{array}{ccc} C(D;S^{2n}) & \longrightarrow & \Omega^2 S^{2n+2} \\ \downarrow & & \downarrow \\ C(M_g;S^{2n}) & \longrightarrow & \operatorname{map}_o(\bar{M}_g;S^{2n+2}) \\ \downarrow & & \downarrow \\ C(M_g,D;S^{2n}) & \longrightarrow & (\Omega S^{2n+2})^{2g} \end{array}$$

where $\operatorname{map}_o$ stands for based maps. The right column is induced by restricting to the 1-section, and is a fibration. The left column is a quasifibration. Since $S^{2n}$ is connected, all three horizontal maps are equivalences, see [M], [B] for details.

The $E_2$-term of the Serre spectral sequence of these (quasi)fibrations is as follows. From the base we have 2g-fold tensor product of

(11) $$H^*\Omega S^{2n+2} = H^*(S^{2n+1} x \Omega S^{4n+3}) = E[z_i] \otimes P[u_i] \quad (i = 1, \ldots 2g),$$

where $|z_i| = 2n+1$ and $|u_i| = 4n+2$. From the fibre we have

(12) $$H^*\Omega^2 S^{2n+2} = H^*(\Omega S^{2n+1} x\ \Omega^2 S^{4n+3}) = H^*(\Omega S^{2n+1} x\ S^{4n+1})$$
$$= P[v] \otimes E[w]\ ,$$

where $|v| = 2n$ and $|w| = 4n+1$. The following determines all differentials in this spectral sequence.

Lemma. The differentials are as follows:

(13) $$d_{2n+1}(v) = 0$$

(14) $$d_{4n+2}(w) = 2z_1 z_2 + 2z_2 z_3 + \ldots + 2z_{2g-1} z_{2g}$$

Proof: Assertion (13) follows from the stable splitting of $C(M_g; S^{2n})$, on [B]. (14) results from symmetries of $M_g$ and of the fibrations (10) which leave d invariant. ■

The lemma implies $E_{4n+3} = E_\infty = H^*C(M_g; S^{2n})$. Furthermore, $E_{4n+3}$ is a tensor product of the polynomial algebra $P[v, u_1, \ldots u_{2g}]$ and the homology module $H_*(E,d)$ of the exterior algebra $E = E[w, z_1, \ldots, z_{2g}]$ with differential d. This proves Theorem A.

4. Homology of E. Let us write $x_i = z_{2i-1}$ and $y_i = z_{2i}$ for $i = 1, \ldots g$. The form $d(w) = 2z_1 z_2 + 2z_2 z_3 + \ldots + 2z_{2g}, z_{2g}$ is equivalent to the standard symplectic form $x_1 y_1 + x_2 y_2 + \ldots + x_g y_g$. The vector space

$E[g] = L[g] \oplus wL[g]$ with $L[g] = E[x_1y_1, \ldots, x_{gi}y_g]$. The differential is zero on the first summand, and sends the second to the first. Hence we regard d as an endomorphism of $L[g]$, given by multiplication with $d(w) = x_1y_1 + \ldots + x_gy_g$.

Let $L_k[g]$ denote the vector subspace spanned by all k-fold products

(15) $z_{i_1}z_{i_2} \ldots z_{i_n}$ with $1 \le i_1 < i_2 < \ldots < i_k \le 2_g$.

Since $d(w)$ is homogeneous of weight 2, we have

(16) $d = d[g] = \bigoplus_{k=0}^{2g} d_k[g], \qquad d_k[g]: L_k[g] \longrightarrow L_{k+2}[g]$ .

The (co)kernelsof $d_k[g]$ is determined by the (co)kernel of $d_l[g-1]$ and $d_l[g-1]^2$ for $l = k, k-1, k-2$. The (co)kernel of $d_l[g-1]^2$ in turn is determined by the (co)kernels of $d_m[g-2]^2$ and $d_m[g-2]^3$ for $m = l, l-1, l-2$. Therefore we will study all powers $d_k[g]^r$ and prove the following (Lefschetz) lemma by simultaneous induction on g, k and r.

<u>Lemma. For</u> $0 \le k \le g$ <u>the differential</u>

$d_k[g]^r : L_k[g] \longrightarrow L_{k+2r}[g]$ <u>is</u>

(17) <u>a monomorphism for</u> $0 \le k < g-r$,

(18) <u>an isomorphism for</u> $k = g-r$

(19) an epimorphism for $g-r < k \le 2g$

Proof: For $\lambda_g = \sum_{i=1}^{g} x_iy_i$ we have $\lambda_g = \lambda_{g-1} + x_gy_g$ and $\lambda_g^r = \lambda_{g-1}^r + r\lambda_{g-1}^{r-1}x_gy_g$, in particular $\lambda_g^g = g!\omega_g$ where $\omega_g = x_1y_1x_2y_2 \ldots x_gy_g$ is the volume element. To facilitate the induction, we decompose $L_k[g]$ further by partitioning the canonical basis elements (15) into four types.

(20) $i_k \le 2g-2$

(21) $i_{k-1} \le 2g-2$ and $i_k = 2g-1$,

(22) $i_{k-1} \leq 2g-2$ and $i_k = 2g$,

(23) $i_{k-1} = 2g-1$ and $i_k = 2g$.

Hence $L_k[g] = L_k[g-1] \oplus L_{k-1}[g-1]x_g \oplus L_{k-1}[g-1]y_g \oplus L_{k-2}[g-1]x_g y_g$. With respect to this decomposition $d_k[g]^r$ has the following matrix form

$$d_k[g]^r = \begin{bmatrix} d_k[g-1]^r & 0 & 0 & rd_k[g-1]^{r-1} \\ 0 & d_{k-1}[g-1]^r & 0 & 0 \\ 0 & 0 & d_{k-1}[g-1]^r & 0 \\ 0 & 0 & 0 & d_{k-2}[g-1]^r \end{bmatrix} = \begin{bmatrix} A & 0 & 0 & A' \\ 0 & B & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & 0 & 0 & C \end{bmatrix} \tag{24}$$

To start the induction consider the case $g = 1$. The only non-zero differential $d_o[1] : L_o[1] \to L_2[1]$ is an isomorphism. For $g \geq 2$ and $k = 0$, $d_o[g]^r$ sends the generator of $L_o[g]$ to $\lambda_g^r$, and thus is monic. Assume the lemma holds for g-1. We distinguish three cases.

Case $k < g-r$: Then A, A', B as well as C in (24) are all monomorphisms by hypothesis. Hence, from $0 = d_k[g]^r(a,b_1,b_2,c) = (A(a), B(b_1), B(b_2), A'(a) + C(c))$ we conclude $a = b_1 = b_2 = 0$, ands so $c = 0$ as well. Thus $d_k[g]^r$ is a monomorphism.

Case $k = g-r$: Here A is an epimorphism, A' and B are isomorphisms, and C is a monomorphism. Assume $0 = d_k[g]^r(a,b_1,b_2,c) = (A(a), B(b_1), A'(a) + C(c))$. First, $b_1 = b_2 = 0$. We now have $A(a) = d_k[g-1]^r a = 0$ and $d_{k-2}[g-1]^r c = -rd_k[g-1]^{r-1}a$; writing this as $d_k[g-1]^r(-ra) = A(-ra)$ $= 0$ . Thus, since $d_{k-2}[g-1]^{r+1}$ is an isomorphism, $c = 0$. Therefore, $-rd_{g-r}[g-1]^{r-1}a = 0$, and $a = 0$ since $d_{g-r}[g-1]^{r-1}$ is an isomorphism. We see that $d_k[g]^r$ is a monomorphism between vector spaces of equal dimensions, hence an isomorphism.

Case $k > g-r$: This time A, A', B, C are epimorphisms. Given $(\bar{a},\bar{b}_1,\bar{b}_2,\bar{c}) \in L_{k+2r}[g]$ we can first find a, $b_1$, $b_2$ satisfying $A(a) = \bar{a}$, $B(b_1) = \bar{b}_1$ and $B(b_2) = \bar{b}_2$. Then we choose c such that $C(c) = \bar{c} - A'(a)$. Hence $d_k[g]^r$ is epimorphic. ∎

The lemma completely determines $H_*(E,d)$ as a vector space over $\mathbb{Q}$. Theorem D now follows.

References


[B] C.-F. Bödigheimer: Stable splittings of mapping spaces. Proc. Seattle (1985), Springer LNM 1286, p. 174-187.

[BCM] C.-F. Bödigheimer, F.R. Cohen, R.J. Milgram: On deleted symmetric products. In preparation.

[BCP] C.-F. Bödigheimer, F.R. Cohen, M. Peim: Mapping spaces and the hyperelliptic mapping class group. In preparation.

[BCT] C.-F. Bödigheimer, F.R. Cohen, L. Taylor: Homology of configuration spaces. To appear in Topology.

[CCKN] F.R. Cohen, R. Cohen, N. Kuhn, J. Neisendorfer: Bundles over configuration spaces. Pac. J. Math. 104 (1983), p. 47-54.

[M] D. McDuff: Configuration spaces of positive and negative particles. Topology 14 (1975), p. 91-107.



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