A New Finestructural Hierarchy for the Constructible Universe*

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Abstract: We present a natural hierarchy for Gödel's model L of constructible sets. The new hierarchy simplifies finestructural arguments. This is demonstrated by a proof of Jensen's Covering Theorem for L.

1. Introduction

The fundamental constructible operation is the formation of a definable subset of a level L_α of Gödel's constructible hierarchy:

$$I(L_\alpha, \varphi, \vec{y}) = \{x \in L_\alpha | L_\alpha \models \varphi[\vec{y}, x]\}$$

where φ is a first-order formula and $\vec{y} \in L_\alpha$. The next level of the hierarchy is then formed as

$$L_{\alpha+1} = \{I(L_\alpha, \varphi, \vec{y}) | \varphi \text{ is a formula}, \vec{y} \in L_\alpha\}.$$

This hierarchy is analysed using Skolem functions

$$S(L_\alpha, \varphi, \vec{y}) = \text{ the least } x \in L_\alpha \text{ such that } L_\alpha \models \varphi[\vec{y}, x]$$

relative to a canonical wellorder $<_{L_\alpha}$ of L_α.

Whereas some principles like the Continuum Hypothesis can be proved in L with "coarse" methods, the proofs of subtle principles like Jensen's $\Box$ require a careful setup and analysis of these structures. In Jensen's *finestructure theory* of the constructible hierarchy [4], first-order definability is split up into iterated Σ_1-definability over appropriate structures. It is essential that

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the constructible operations are definable by $\in$-formulas of low complexity which has to be shown with considerable effort.
This article presents an approach which aims at reducing these complications: the new hierarchy is built with a definability notion of the order of Σ_1-definability; the corresponding constructible operations and notions are incorporated into the language as basic symbols.
Combinatorial applications of finestructure theory are based on the uniform behaviour of hulls under finestructural Skolem functions. Constellations in the constructible universe can be captured by finestructural hulls; the *condensation lemma* controls the shape of such hulls since they have to be isomorphic to levels of the hierarchy.
The hulls formed in the new finestructure satisfy laws known from the classical theory, and they are therefore adequate for finestructural arguments. We demonstrate this by a proof of the Jensen Covering Theorem for L which is simpler than the original proof in [1] since it need not distinguish cases according to the various complexities of definitions.
The article is structured as follows: Sections 2 - 5 introduce the basic notions of the *fine hierarchy*; we prove condensation and an analogous result about directed limits. Section 6 transfers Jensen's *upward extensions of embeddings technique* to the fine hierarchy; the method basically defines an ultrapower by an extender, so we call the resulting map an *extension*. In section 7 we construct strong maps with *wellfounded* extensions. The techniques are put together in section 8 for a proof of the Covering Theorem.
The techniques of the proof of the Covering Theorem can be used in *core model theory*. In particular, fine ultrapowers and extensions can be formed by the techniques of section 6.

2. The Fine Hierarchy

We approximate the constructible universe by a hierarchy $(\mathcal{F}_\alpha)_{\alpha\in\mathrm{On}}$ of structures $\mathcal{F}_\alpha = (F_\alpha, I\restriction F_\alpha, S\restriction F_\alpha, \in, <\restriction F_\alpha)$. Each F_α is a transitive set and $\bigcup_{\alpha\in\mathrm{On}} F_\alpha = L$. The functions $I\restriction F_\alpha$ and $S\restriction F_\alpha$ are the restrictions to $F_\alpha^{<\omega}$ of a global *interpretation function* I and a global *Skolem function* S defined on $L^{<\omega}$. The relation $<\restriction F_\alpha$ is the restriction of a ternary *guarded constructible wellorder* to F_α^3. We shall usually write $(F_\alpha, I, S, \in, <)$ instead of $\mathcal{F}_\alpha = (F_\alpha, I\restriction F_\alpha, S\restriction F_\alpha, \in, <\restriction F_\alpha)$. We first define a language adequate for the structures $\mathcal{F}_\alpha$.

2.1. Definition. Let $\mathcal{L}$ be the first-order language with the following components:

- variable symbols $\dot{v}_n$ for $n < \omega$;
- logical symbols $\dot{=}$ (equality), $\dot{\wedge}$ (conjunction), $\dot{\neg}$ (negation), $\dot{\exists}$ (existential quantification), (,) (brackets);
- function symbols $\dot{I}$ (interpretation), $\dot{S}$ (Skolem function) of variable finite arity;
- a binary relation symbol $\dot{\in}$ (set-membership) and a ternary relation symbol $\dot{<}$ (guarded wellorder).

The syntax and semantics of $\mathcal{L}$ are defined as usual. The variable arity of $\dot{I}$ and $\dot{S}$ is handled by bracketing: if $n < \omega$ and $t_0, \ldots, t_{n-1}$ are $\mathcal{L}$-terms then $\dot{I}(t_0, \ldots, t_{n-1})$ and $\dot{S}(t_0, \ldots, t_{n-1})$ are $\mathcal{L}$-terms. If t_0, t_1, t_2 are $\mathcal{L}$-terms then $t_0 \dot{\in} t_1$ and $t_0 \dot{<}_{t_1} t_2$ are atomic $\mathcal{L}$-formulas. We denote the set of first-order $\mathcal{L}$-formulas simply by $\mathcal{L}$.

We assume that $\mathcal{L}$ is Gödelised conveniently: the set of Gödelised formulas satisfies $\mathcal{L} \subseteq \{V_n \mid n < \omega\}$ (this technical condition will be used in used in Proposition 3.6); the usual syntactical operations of $\mathcal{L}$ are recursively definable over V_ω. This includes the simultaneous substitution $\varphi\frac{\vec{t}}{\vec{w}}$ of terms $\vec{t}$ for variables $\vec{w}$ in φ. The notation $\varphi(\dot{v}_0, \ldots, \dot{v}_{n-1})$ implies that the free variables of φ are contained in $\{\dot{v}_0, \ldots, \dot{v}_{n-1}\}$. By $\mathcal{L}_0$ we denote the collection of quantifier-free formulas of $\mathcal{L}$.

The language $\mathcal{L}$ is interpreted in $\mathcal{L}$-structures in the obvious way. If $\mathcal{A} = (A, \ldots)$ is an $\mathcal{L}$-structure, $\varphi(\dot{v}_0, \ldots, \dot{v}_{n-1}) \in \mathcal{L}$ and $a_0, \ldots, a_{n-1} \in A$ then $\mathcal{A} \models \varphi[a_0, \ldots, a_{n-1}]$ means that $\mathcal{A}$ is a model of φ under the variable assignment $\dot{v}_i \mapsto a_i$ for $i < n$. The $\mathcal{F}_\alpha$-hierarchy is defined by iterated $\mathcal{L}_0$-definability:

2.2. Definition. The *fine hierarchy* consists of *fine levels* $\mathcal{F}_\alpha = (F_\alpha, I \upharpoonright F_\alpha, S \upharpoonright F_\alpha, \in, < \upharpoonright F_\alpha)$ which are defined by recursion on $\alpha \in \mathrm{On}$:

For $\alpha \leq \omega$ let $F_\alpha = V_\alpha$ and $\forall \vec{x} \in F_\alpha : I(\vec{x}) = S(\vec{x}) = 0$. Let $<_{F_\omega}$ be a binary relation which wellorders $F_\omega = V_\omega$ in ordertype ω and which extends the $\in$-relation. Define the ternary relation $< \upharpoonright F_\omega$ by: $x <_y z$ iff ($y = F_n$ for some $n < \omega$, $x, z \in y$ and $x <_{F_\omega} z$). This defines the structures $\mathcal{F}_\alpha$ for $\alpha \leq \omega$.

Assume that $\alpha \geq \omega$ and that $\mathcal{F}_\alpha = (F_\alpha, I \upharpoonright F_\alpha, S \upharpoonright F_\alpha, \in, < \upharpoonright F_\alpha)$ has been defined. For $\varphi(\dot{v}_0, \ldots, \dot{v}_n) \in \mathcal{L}_0$ and $x_0, \ldots, x_{n-1} \in F_\alpha$ set

$$(*) \qquad I(F_\alpha, \varphi, x_0, \ldots, x_{n-1}) = \{x_n \in F_\alpha \mid \mathcal{F}_\alpha \models \varphi[x_0, \ldots, x_n]\}.$$

We say that $(F_\alpha, \varphi, x_0, \ldots, x_{n-1})$ is a *name* for its *interpretation* $I(F_\alpha, \varphi, x_0, \ldots, x_{n-1})$ in the fine hierarchy.

The next fine level is defined as

$$F_{\alpha+1} = \{I(F_\alpha, \varphi, x_0 \dots x_{n-1}) \,|\, \varphi(\dot v_0 \dots \dot v_n) \in \mathcal{L}_0, x_0 \dots x_{n-1} \in F_\alpha,\, n < \omega\}.$$

To define $I \restriction F_{\alpha+1}$ we only need to define $I(\vec z)$ for "new" vectors $\vec z \in (F_{\alpha+1})^{<\omega} \setminus F_\alpha^{<\omega}$. We have already made certain assignments in $(*)$; in all other cases set $I(\vec z) = 0$. Define the relation $<_{F_\alpha}$ endextending $<_{F_\omega}$: for $x, y \in F_\alpha \setminus F_\omega$ set $x <_{F_\alpha} y$ iff there is a name $(F_\beta, \varphi, x_0, \dots, x_{m-1})$ for x such that every name $(F_\gamma, \psi, y_0, \dots, y_{n-1})$ for y is lexicographically greater than $(F_\beta, \varphi, x_0, \dots, x_{m-1})$, where the first two coordinates are wellordered by $\in$ and the further coordinates are wellordered by $<_{F_\gamma}$. The ternary relation $<\restriction F_{\alpha+1}$ is then defined by: $x <_y z$ iff $y = F_\nu$ for some $\nu \le \alpha$ and $x <_{F_\nu} z$. The Skolem function S finds witnesses of existential statements. Again we only need to define $S(\vec z)$ for $\vec z \in (F_{\alpha+1})^{<\omega} \setminus F_\alpha^{<\omega}$. We set $S(\vec z) = 0$ except when $\vec z = F_\alpha, \varphi(\dot v_0, \dots, \dot v_n), x_0, \dots, x_{m-1}$, where $\varphi \in \mathcal{L}_0$, $x_0, \dots, x_{m-1} \in F_\alpha$, $m \le n$, and there exist $x_m, x_{m+1}, \dots, x_n \in F_\alpha$ such that: $\mathcal{F}_\alpha \models \varphi[x_0, \dots, x_{m-1}, x_m, x_{m+1}, \dots, x_n]$; in this case let $S(\vec z)$ be such an x_m minimal with respect to the wellfounded relation $<_{F_\alpha}$.

This defines $\mathcal{F}_{\alpha+1} = (F_{\alpha+1}, I \restriction F_{\alpha+1}, S \restriction F_{\alpha+1}, \in, <\restriction F_{\alpha+1})$.

Assume that $\lambda > \omega$ is a limit ordinal and that $\mathcal{F}_\alpha$ is defined for $\alpha < \lambda$. Then let $F_\lambda = \bigcup_{\alpha<\lambda} F_\alpha$, which determines $\mathcal{F}_\lambda = (F_\lambda, I \restriction F_\lambda, S \restriction F_\lambda, \in, <\restriction F_\lambda)$.

The fine hierarchy satisfies basic hierarchical properties some of which were assumed tacitely in the previous definition:

2.3. Proposition. For every $\gamma \in \mathrm{On}$:

(a) $\alpha \le \gamma \to F_\alpha \subseteq F_\gamma$;

(b) $\alpha < \gamma \to F_\alpha \in F_\gamma$;

(c) F_γ is transitive.

Proof. By simultaneous induction on γ. Assume that (a) – (c) hold for $\beta < \gamma$. Then they trivially hold at γ if $\gamma \le \omega$ or γ is a limit ordinal.

Consider the remaining case $\gamma = \beta + 1$, $\beta \ge \omega$.

(a) It suffices to show that $F_\beta \subseteq F_{\beta+1}$. Let $z \in F_\beta$. Since F_β is transitive, $z = \{x \in F_\beta \,|\, x \in z\} = \{x \in F_\beta \,|\, \mathcal{F}_\beta \models (\dot v_1 \dot\in \dot v_0)[z, x]\} \in F_{\beta+1}$.

(b) By (a), it suffices to show that $F_\beta = \{x \in F_\beta \,|\, \mathcal{F}_\beta \models (\dot v_0 \dot= \dot v_0)[x]\} \in F_{\beta+1}$.

(c) Let $a \in F_{\beta+1}$. Then $a \subseteq F_\beta \subseteq F_{\beta+1}$, and $a \subseteq F_{\beta+1}$. **qed**

First-order definability can be emulated in $\mathcal{S}_0$:

2.4. Proposition. For every $\in$-formula $\varphi(\dot{v}_0, \ldots, \dot{v}_{m-1})$ one can uniformly define a quantifier-free formula $\varphi^*(\dot{v}_0, \ldots, \dot{v}_{m-1}, \dot{v}_m, \ldots, \dot{v}_{m+k-1}) \in \mathcal{L}_0$ such that for all $\alpha \geq \omega$ and for all $a_0, \ldots, a_{m-1} \in F_\alpha$:

$$(F_\alpha, \in) \models \varphi[a_0, \ldots, a_{m-1}] \quad \text{iff} \quad \mathcal{F}_{\alpha+k} \models \varphi^*[a_0, \ldots, a_{m-1}, F_\alpha, \ldots, F_{\alpha+(k-1)}].$$

Proof. By induction on the complexity of φ. For atomic φ let $\varphi^* = \varphi$, for $\varphi = \dot{\neg}\varphi_0$ let $\varphi^* = \dot{\neg}\varphi_0^*$, and for $\varphi = \varphi_0 \dot{\wedge} \varphi_1$ set $\varphi^* = \varphi_0^* \dot{\wedge} \varphi_1^*$. The only nontrivial case is the existential one. Consider φ, φ^* satisfying the proposition. Then: $(F_\alpha, \in) \models \dot{\exists}\dot{v}_j \varphi[a_0, \ldots, a_{m-1}]$

$\iff \exists b_j \in F_\alpha \; (F_\alpha, \in) \models \varphi[a_0, \ldots, a_{j-1}, b_j, a_{j+1}, \ldots, a_{m-1}]$

$\iff \exists b_j \in F_\alpha \; \mathcal{F}_{\alpha+k} \models \varphi^*[a_0, \ldots, b_j, \ldots, a_{m-1}, F_\alpha, \ldots, F_{\alpha+(k-1)}]$

$\iff \{x_{m+k} \in F_\alpha \,|\, \mathcal{F}_{\alpha+k} \models (\varphi^* \frac{\dot{v}_{m+k}}{\dot{v}_j})[a_0, \ldots, a_{m-1}, F_\alpha, \ldots,$
$F_{\alpha+(k-1)}, x_{m+k}]\} \neq \emptyset$

$\iff \{x_{m+k} \in F_{\alpha+k} \,|\, \mathcal{F}_{\alpha+k} \models (\dot{v}_{m+k} \dot{\in} \dot{v}_m \dot{\wedge} \varphi^* \frac{\dot{v}_{m+k}}{\dot{v}_j})$
$[a_0, \ldots, F_\alpha, \ldots, x_{m+k}]\} \neq \emptyset$

$\iff I(F_{\alpha+k}, (\dot{v}_{m+k} \dot{\in} \dot{v}_m \dot{\wedge} \varphi^* \frac{\dot{v}_{m+k}}{\dot{v}_j}), a_0, \ldots, a_{m-1}, F_\alpha, \ldots, F_{\alpha+(k-1)}) \neq \emptyset.$

The right-hand side is expressible quantifier-free in $\mathcal{F}_{\alpha+k+1}$, which yields a formula $(\exists \dot{v}_j \varphi)^*$ as required. **qed**

By the following theorem we have defined a hierarchy for the constructible universe:

2.5. Theorem. $\bigcup_{\alpha \in \mathrm{On}} F_\alpha = L$.

Proof. ($\subseteq$) holds since the definition of the fine hierarchy can be carried out absolutely inside the inner model L.

For ($\supseteq$) set $F_\infty = \bigcup_{\alpha \in \mathrm{On}} F_\alpha$. It suffices to show that F_∞ is an inner model of set theory since L is the $\subseteq$-smallest inner model. By a variant of Theorem 13.9 of [3] it is enough to check the following three facts:

(1) F_∞ is transitive;

this holds by 2.3(c).

(2) F_∞ is closed with respect to first-order definability, i.e., for all $\in$-formulas $\varphi(v_0, \ldots, v_{m-1})$ and $a_1, \ldots, a_{m-1}, z \in F_\infty$ we have $\{a_0 \in z \,|\, (z, \in) \models \varphi[a_0, \ldots, a_{m-1}]\} \in F_\infty$.

Proof. Let $a_1, \ldots, a_{m-1}, z \in F_\alpha$ and let $\varphi^{\dot{v}_m}$ be the formula φ with all quantifiers restricted to the new variable $\dot{v}_m$. Then:

$$\begin{aligned}
&\{a_0 \in z \mid (z, \in) \models \varphi[a_0, \ldots, a_{m-1}]\}\\
&= \{a_0 \in z \mid (F_\alpha, \in) \models \varphi^{\dot{v}_m}[a_0, \ldots, a_{m-1}, z]\}\\
&= \{a_0 \in F_{\alpha+k} \mid \mathcal{F}_{\alpha+k} \models (\dot{v}_0 \dot{\in} \dot{v}_m \dot{\wedge} \varphi^{\dot{v}_m})^*\\
&\qquad\qquad [a_0, \ldots, a_{m-1}, z, F_\alpha, \ldots, F_{\alpha+(k-1)}]\},\\
&\text{by 2.4.}\\
&= \{a_0 \in F_{\alpha+k} \mid \mathcal{F}_{\alpha+k} \models (\dot{v}_0 \dot{\in} \dot{v}_m \dot{\wedge} \varphi^{\dot{v}_m})^* \frac{\dot{v}_{m+k+1}}{\dot{v}_0}[0, a_1, \ldots, z, F_\alpha, \ldots, a_0]\}\\
&\in F_{\alpha+k+1}
\end{aligned}$$

as required. qed(2)

(3) F_∞ is almost universal, i.e., $\forall y \subseteq F_\infty \exists z \in F_\infty : y \subseteq z$.

Proof. If $y \subseteq F_\infty$, then $y \subseteq F_\alpha$ for some α. By 2.3.(b) $F_\alpha \in F_{\alpha+1}$ and so $z = F_\alpha$ satisfies the claim. **qed**

Remark. A more detailed analysis (see [5]) shows that the $\mathcal{F}$-hierarchy is actually a refinement of Jensen's J_α-hierarchy: $\forall \nu \in \mathrm{On}\ \ J_\nu = F_{\omega\nu}$.

The constructible sets are generated by the I-function from the levels F_α:

2.6. Proposition. If $x \in F_{\gamma+1}$, $\gamma \geq \omega$ then there are $\alpha_0, \ldots, \alpha_{m-1} \leq \gamma$ such that $x = I(F_{\alpha_0}, \ldots, F_{\alpha_{m-1}})$.

Proof. We show the proposition by induction on γ. So assume the property for $y \in F_{\beta+1}$ with $\beta < \gamma$. By definition of $F_{\gamma+1}$, $x = I(F_\gamma, \varphi, y_0, \ldots, y_{n-1})$ for some $\varphi \in \mathcal{S}_0$ and $y_0, \ldots, y_{n-1} \in F_\gamma$.
Let $y_i = I(F_{\alpha_0^i}, \ldots, F_{\alpha^i_{m(i)-1}})$ for $i < n$ according to the inductive assumption. Then

$$\begin{aligned}
x &= I(F_\gamma, \varphi, I(F_{\alpha_0^0}, \ldots, F_{\alpha^0_{m(0)-1}}), \ldots, I(F_{\alpha_0^{n-1}}, \ldots, F_{\alpha^{n-1}_{m(n)-1}}))\\
&= I(F_\gamma, \psi, F_{\alpha_0^0}, \ldots, F_{\alpha^0_{m(0)-1}} \ldots, F_{\alpha_0^{n-1}}, \ldots, F_{\alpha^{n-1}_{m(n)-1}}),
\end{aligned}$$

where

$$\begin{aligned}
&\psi(\dot{v}_0^0, \ldots, \dot{v}^0_{m(0)-1}, \ldots, \dot{v}_0^{n-1}, \ldots, \dot{v}^{n-1}_{m(n)-1}, \dot{v}_r)\\
&= \varphi \frac{\dot{I}(\dot{v}_0^0, \ldots, \dot{v}^0_{m(0)-1}), \ldots, \dot{I}(\dot{v}_0^{n-1}, \ldots, \dot{v}^{n-1}_{m(n)-1}), \dot{v}_r}{\dot{v}_0, \dot{v}_1, \ldots, \dot{v}_n}
\end{aligned}$$

and $\dot{v}_0^0, \ldots, \dot{v}_{m(0)-1}^0, \ldots, \dot{v}_0^{n-1}, \ldots, \dot{v}_{m(n)-1}^{n-1}, \dot{v}_r$ is the sequence of variables up to and including $\dot{v}_r$. **qed**

3. Constructible Hulls and Maps

The functions I and S are the basic *constructible operations*. We consider some related "algebraic notions".

3.1. Definition. A set or class $Z \subseteq L$ is *constructibly closed* if $F_\omega \subseteq Z$ and Z is closed with respect to the operations I and S. For $X \subseteq L$ let $\mathcal{F}(X)$ be the *constructible hull* of X, i.e., the $\subseteq$-smallest constructibly closed superset of X. Note that all fine levels F_α are constructibly closed.

3.2. Definition. A map $\sigma \colon \mathcal{A} \to \mathcal{B}$ of $\mathcal{S}$-structures is called *fine*, if it preserves all quantifier-free $\mathcal{L}$-formulas and if $\sigma \restriction F_\omega = \mathrm{id} \restriction F_\omega$. If σ is an isomorphism, we call it a *fine isomorphism* and write $\sigma \colon \mathcal{A} \tilde{\to} \mathcal{B}$.

We are particularly interested in fine maps between constructible levels $\mathcal{F}_\alpha$. In some constructions one actually obtains stronger closure properties with respect to the functions $I(F_\alpha, \_, \_)$ and $S(F_\alpha, \_, \_)$.

3.3. Definition. A set $Z \subseteq \mathcal{F}_\alpha$ is *constructibly closed up to* $\mathcal{F}_\alpha$, if Z is constructibly closed and:

(a) $\forall \varphi \in \mathcal{L}_0 \forall \vec{x} \in Z\ (I(F_\alpha, \varphi, \vec{x}) \in F_\alpha \to I(F_\alpha, \varphi, \vec{x}) \in Z)$;

(b) $\forall \varphi \in \mathcal{L}_0 \forall \vec{x} \in Z\ S(F_\alpha, \varphi, \vec{x}) \in Z$.

Obviously:

3.4. Proposition. If Z is constructibly closed and $F_\alpha \in Z$ then $Z \cap F_\alpha$ is constructibly closed up to $\mathcal{F}_\alpha$.

3.5. Lemma. Let $Z \subseteq \mathcal{F}_\alpha$ be constructibly closed up to $\mathcal{F}_\alpha$. Then Z is existentially closed inside $\mathcal{F}_\alpha$, i.e., for all $\varphi(\dot{v}_0, \ldots, \dot{v}_{m-1}, \dot{v}_m, \ldots, \dot{v}_n) \in \mathcal{L}_0$: $\forall a_0, \ldots, a_{m-1} \in Z : ((Z, I, S, \in, <) \models \exists \dot{v}_m \ldots \dot{v}_n \varphi[\vec{a}]$ iff $\mathcal{F}_\alpha \models \exists \dot{v}_m \ldots \dot{v}_n \varphi[\vec{a}])$.

Proof. Assume that $\mathcal{F}_\alpha \models \exists \dot{v}_m \ldots \dot{v}_n \varphi[\vec{a}]$. It suffices to show by induction on $i = m, \ldots, n$ that

$$\exists \dot{v}_m \in Z \ldots \exists \dot{v}_{i-1} \in Z \exists \dot{v}_i \in F_\alpha \ldots \exists \dot{v}_n \in F_\alpha\ \varphi[\vec{a}].$$

For the inductive step take $b_m, \dots, b_{i-1} \in Z$ such that

$$\exists \dot{v}_i \in F_\alpha \dots \exists \dot{v}_n \in F_\alpha \; \varphi[\vec{a}, \vec{b}].$$

Set $b_i = S(F_\alpha, \varphi, \vec{a}, \vec{b})$. Then by the definition of S

$$\exists \dot{v}_{i+1} \in F_\alpha \dots \exists \dot{v}_n \in F_\alpha \; \varphi[\vec{a}, \vec{b}, b_i].$$

Since Z is closed with respect to the function $S(F_\alpha, \_, \_)$, we get $b_i \in Z$.**qed**

3.6. Definition. A map $\sigma \colon \mathcal{F}_\alpha \to \mathcal{F}_\beta$ is *fine up to* $\mathcal{F}_\beta$ if σ is fine and its range $\sigma'' F_\alpha$ is constructibly closed up to $\mathcal{F}_\beta$.

Then 3.5 implies immediately:

3.7. Lemma. If $\sigma \colon \mathcal{F}_\alpha \to \mathcal{F}_\beta$ is fine up to $\mathcal{F}_\beta$ then σ is elementary for existential $\mathcal{L}$-formulae, i.e., for $\varphi(\dot{v}_0, \dots, \dot{v}_{m-1}, \dot{v}_m, \dots, \dot{v}_n) \in \mathcal{L}_0$

$$\forall a_0, \dots, a_{m-1} \in F_\alpha \; \mathcal{F}_\alpha \models \exists \dot{v}_m, \dots, \dot{v}_n \; \varphi[\vec{a}] \;\text{ iff }\; \mathcal{F}_\beta \models \exists \dot{v}_m, \dots, \dot{v}_n \; \varphi[\sigma(\vec{a})]).$$

These maps allow liftings to the next $\mathcal{F}$-levels:

3.8. Lemma. Let $\sigma \colon \mathcal{F}_\alpha \to \mathcal{F}_\beta$ be fine up to $\mathcal{F}_\beta$. Then there is a uniquely determined fine lifting $\sigma^+ \colon \mathcal{F}_{\alpha+1} \to \mathcal{F}_{\beta+1}$ such that $\sigma^+ \supseteq \sigma$ and $\sigma^+(F_\alpha) = F_\beta$.

Proof.

(1) For $\varphi(\dot{v}_0, \dots, \dot{v}_m) \in \mathcal{L}_0$, $\vec{x} = x_0, \dots, x_{m-1} \in F_\alpha$ and $\psi(\dot{v}_0, \dots, \dot{v}_n) \in \mathcal{L}_0$, $\vec{y} = y_0, \dots, y_{n-1} \in F_\alpha$:
$I(F_\alpha, \varphi, \vec{x}) = I(F_\alpha, \psi, \vec{y})$ iff $I(F_\beta, \varphi, \sigma(\vec{x})) = I(F_\beta, \psi, \sigma(\vec{y}))$.

Proof. Assuming that $m \le n$:

$$\begin{aligned} & I(F_\alpha, \varphi, \vec{x}) \neq I(F_\alpha, \psi, \vec{y}) \\ \iff & \mathcal{F}_\alpha \models \dot{\exists} \dot{v}_m (\dot{\neg} \varphi \dot{\leftrightarrow} \psi \frac{\dot{v}_{m+1} \dots \dot{v}_{m+n} \dot{v}_m}{\dot{v}_0 \dots \dot{v}_{n-1} \dot{v}_n})[\vec{x}, 0, \vec{y}] \\ \iff & \mathcal{F}_\beta \models \dot{\exists} \dot{v}_m (\dot{\neg} \varphi \dot{\leftrightarrow} \psi \frac{\dot{v}_{m+1} \dots \dot{v}_{m+n} \dot{v}_m}{\dot{v}_0 \dots \dot{v}_{n-1} \dot{v}_n})[\sigma(\vec{x}), 0, \sigma(\vec{y})], \quad \text{by 3.7,} \\ \iff & I(F_\beta, \varphi, \sigma(\vec{x})) \neq I(F_\beta, \psi, \sigma(\vec{y})). \end{aligned}$$

qed(1)

So we can define an injective map $\sigma^+ \colon F_{\alpha+1} \to F_{\beta+1}$ by $\sigma^+(I(F_\alpha, \varphi, \vec{x})) = I(F_\beta, \varphi, \sigma(\vec{x}))$.

(2) $\sigma^+ \supseteq \sigma$ and $\sigma^+(F_\alpha) = F_\beta$.

Proof. Let $x \in F_\alpha$. Then $x = I(F_\alpha, \dot{v}_1 \dot{\in} \dot{v}_0, x)$ and
$\sigma^+(x) = I(F_\beta, \dot{v}_1 \dot{\in} \dot{v}_0, \sigma(x)) = \sigma(x)$.
Furthermore $\sigma^+(F_\alpha) = \sigma^+(I(F_\alpha, \dot{v}_0 \doteq \dot{v}_0)) = I(F_\beta, \dot{v}_0 \doteq \dot{v}_0) = F_\beta$. qed(2)

(3) $\forall z \in F_{\alpha+1}$: $z \in F_\alpha$ iff $\sigma^+(z) \in F_\beta$.

Proof. Let $z = I(F_\alpha, \varphi, \vec{x})$ and assume that $\sigma^+(z) = I(F_\beta, \varphi, \sigma(\vec{x})) \in F_\beta$. Since σ is fine up to $\mathcal{F}_\beta$, $\sigma^+(z) \in \text{range}(\sigma)$. Let $\sigma^+(z) = \sigma(y)$ for some $y \in F_\alpha$. Then $\sigma^+(z) = \sigma^+(y)$ and $z = y \in F_\alpha$ as required. qed(3)

(4) σ^+ preserves I, i.e., for all $\vec{z} \in F_{\alpha+1}$: $\sigma^+(I(\vec{z})) = I(\sigma^+(\vec{z}))$.

Proof. Clear for $\vec{z} \in F_\alpha$, since σ is fine. If $\vec{z}$ is of the form $\vec{z} = F_\alpha, \varphi, \vec{x}$, $\varphi \in \mathcal{L}_0$, $\vec{x} \in F_\alpha$, then the preservation follows directly from the definition of σ^+. In all other cases we set $I(\vec{z}) = 0$ by default and then, by $\sigma^+(F_\alpha) = F_\beta$, $\sigma^+ \restriction F_\omega = \text{id} \restriction F_\omega$ and by (3): $I(\sigma^+(\vec{z})) = 0$ by default. qed(4)

(5) $\forall y, z \in F_\alpha$ ($y <_{F_\alpha} z$ iff $\sigma(y) <_{F_\beta} \sigma(z)$).

Proof. The relation $<_{F_\alpha}$ is defined via the lexicographical order of names. Let $y{=}I(F_\gamma, \varphi, \vec{x})$, where $(F_\gamma, \varphi, \vec{x})$ is the lexicographically minimal name for $y \in F_\alpha$. Then $\sigma(y){=}I(\sigma(F_\gamma), \varphi, \sigma(\vec{x}))$, and we claim that $(\sigma(F_\gamma), \varphi, \sigma(\vec{x}))$ is the lexicographically minimal name for $\sigma(y)$. If not, then there is a formula $\psi \in \mathcal{L}_0$ such that

$$\begin{aligned}\mathcal{F}_\beta \models\ & \exists v_{n+1} \exists v_0 \ldots v_{n-1} \\ & \sigma(y) = I(v_{n+1}, \psi, v_0, \ldots, v_{n-1}) \wedge v_{n+1} = I(v_{n+1}, \dot{v_0} \doteq \dot{v_0}) \wedge \\ & (v_{n+1}, \psi, v_0, \ldots, v_{n-1}) \text{ is lexicographically smaller than} \\ & (\sigma(F_\gamma), \varphi, \sigma(\vec{x})) \text{ with respect to the wellordering } <_{\sigma(F_\gamma)}.\end{aligned}$$

By 3.5, this existential formula can be pulled back to $\mathcal{F}_\alpha$ by σ which contradicts the minimal choice of the name for y.

For $y, z \in F_\alpha$ the preservation of minimal names yields the following equivalences:
$y <_{F_\alpha} z$
iff the minimal name for y is lexicographically smaller than the minimal name for z
iff the minimal name for $\sigma(y)$ is lexicographically smaller than the minimal name for $\sigma(z)$
iff $\sigma(y) <_{F_\beta} \sigma(z)$. qed(5)
Property (5) implies readily that

(6) σ preserves the ternary relation $<$.

(7) σ^+ preserves S.

Proof. Clear for $\vec{z} \in F_\alpha$. We only have to consider the non-default case when $\vec{z}$ is of the form $\vec{z} = F_\alpha, \varphi, x_0, \ldots, x_{m-1}$, where $\varphi(\dot{v}_0, \ldots, \dot{v}_n) \in \mathcal{L}_0$, $m \leq n$, $x_0, \ldots, x_{m-1} \in F_\alpha$.
$S(\vec{z})$ is defined non-trivially
iff $\mathcal{F}_\alpha \models \dot{\exists}\dot{v}_m \ldots \dot{v}_n\ \varphi[\vec{x}]$
iff $\mathcal{F}_\beta \models \dot{\exists}\dot{v}_m \ldots \dot{v}_n\ \varphi[\sigma(\vec{x})]$
iff $N(\sigma^+(\vec{z}))$ is defined non-trivially,
using the existential elementarity of σ (3.7).

So assume that $z = S(F_\alpha, \varphi, \vec{x})$ is defined non-trivially.

$$\mathcal{F}_\alpha \models \dot{\exists}\dot{v}_{m+1} \ldots \dot{v}_n\ \varphi[\vec{x}, z],$$
$$\mathcal{F}_\beta \models \dot{\exists}\dot{v}_{m+1} \ldots \dot{v}_n\ \varphi[\sigma(\vec{x}), \sigma(z)],$$

and then $S(F_\beta, \varphi, \sigma(\vec{x})) = \sigma(z)$ or $S(F_\beta, \varphi, \sigma(\vec{x})) <_{F_\beta} \sigma(z)$. We have to exclude the second possibility. So assume $S(F_\beta, \varphi, \sigma(\vec{x})) <_{F_\beta} \sigma(z)$ and work for a contradiction: $\mathcal{F}_\beta \models \dot{\exists}\dot{v}_m \ldots \dot{v}_n\ \varphi[\sigma(\vec{x})]$. Since $\sigma'' F_\alpha$ is closed under $S(F_\beta, \_, \_)$ there are $x_m, \ldots, x_n \in F_\alpha$ such that

$$\mathcal{F}_\beta \models \varphi[\sigma(\vec{x}), \sigma(x_m), \ldots, \sigma(x_n)] \text{ and } \sigma(x_m) <_{F_\beta} \sigma(z).$$

Since σ is fine and preserves $<$ this implies:

$$\mathcal{F}_\alpha \models \varphi[\vec{x}, x_m, \ldots, x_n] \text{ and } x_m <_{F_\alpha} z,$$

which contradicts the minimal choice of $z = S(F_\alpha, \varphi, \vec{x})$. qed(8)

(8) σ^+ preserves $\in$, i.e., for $y, z \in F_{\alpha+1}$: $y \in z$ iff $\sigma^+(y) \in \sigma^+(z)$.

Proof. Let $z = I(F_\alpha, \varphi, \vec{x})$. Then

$$\begin{aligned} y \in z &\iff y \in F_\alpha \text{ and } \mathcal{F}_\alpha \models \varphi[\vec{x}, y] \\ &\iff \sigma^+(y) \in F_\beta \text{ and } \mathcal{F}_\beta \models \varphi[\sigma(\vec{x}), \sigma^+(y)] \\ &\iff \sigma^+(y) \in \sigma^+(z). \end{aligned}$$

qed

4. Condensation

Condensation lemmas are the principal tools of constructibility theory:

4.1. Lemma. Let Z be constructibly closed with the additional closure property $Z = I[Z^{<\omega}]$, i.e., every element of Z has a name in Z. Then there is a unique $\beta \in \mathrm{On}$ and a unique fine isomorphism $\pi\colon \mathcal{F}_\beta \tilde{\to} (Z, I, S, \in, <)$. π is called the *condensation map* of Z.

Proof. If β, π like this exist then π is the inverse of the Mostowski collapse of Z and hence uniquely determined. We prove the existence of β, π for $Z \subseteq F_\delta$ by induction on $\delta \leq \infty$. If $\delta = \omega$, then $Z = F_\omega$ and we can take π to be the identity on F_ω.

Assume $\delta > \omega$ and that the property holds for all $\gamma < \delta$: for $\gamma < \delta$ there is a unique $\beta(\gamma) \in \mathrm{On}$ and a unique fine isomorphism $\pi_\gamma\colon \mathcal{F}_{\beta(\gamma)} \tilde{\to} (Z \cap F_\gamma, I, S, \in, <)$. The Mostowski collapses of the $Z \cap F_\gamma$ cohere nicely, and so $\gamma \leq \gamma' < \delta \Rightarrow \beta(\gamma) \leq \beta(\gamma')$ and $\pi_\gamma \subseteq \pi'_\gamma$.

In case δ is a limit ordinal $\leq \infty$, let $\beta = \bigcup_{\gamma<\delta} \beta(\gamma)$ and $\pi = \bigcup_{\gamma<\delta} \pi_\gamma$; then $\pi\colon \mathcal{F}_\beta \tilde{\to} (Z, I, S, \in, <)$ is as required.

Finally consider the successor case $\delta = \gamma + 1$. If $Z \subseteq F_\gamma$, we are done by the inductive assumption. So let $z \in Z \cap (F_\delta \setminus F_\gamma) \neq \emptyset$.

By the additional closure property, z has a name in Z which can only be of the form $(F_\gamma, \_, \_)$; hence $F_\gamma \in Z$. By 3.4 $Z \cap F_\gamma$ is constructibly closed up to $\mathcal{F}_\gamma$. Therefore the map $\pi_\gamma\colon \mathcal{F}_{\beta(\gamma)} \to (Z \cap F_\gamma, I, N, \in, <)$ is constructibly closed up to $\mathcal{F}_\gamma$. By 3.8, there is a fine map $\pi^+_\gamma\colon \mathcal{F}_{\beta(\gamma)+1} \to \mathcal{F}_{\gamma+1} = \mathcal{F}_\delta$, $\pi^+_\gamma \supseteq \pi_\gamma$, $\pi^+_\gamma(F_{\beta(\gamma)}) = F\gamma$ and it suffices to show that $Z = \mathrm{range}(\pi^+_\gamma)$.

By definition of π^+_γ, every element of $\mathrm{range}(\pi^+_\gamma)$ is of the form $z = I(F_\gamma, \varphi, \pi_\gamma(\vec{x}))$, and then $z \in Z$ since $F_\gamma, \pi_\gamma(\vec{x}) \in Z$. Conversely consider $z \in Z$. If $z \in Z \cap F_\gamma$ then $z \in \mathrm{range}(\pi_\gamma) \subseteq \mathrm{range}(\pi^+_\gamma)$. If $z \in Z \setminus F_\gamma$, then the closure property $Z = I[Z^{<\omega}]$ implies that $z = I(F_\gamma, \varphi, \vec{y})$ for some $\vec{y} \in Z \cap F_\gamma$. $\vec{y} = \pi_\gamma(\vec{x})$ for some $\vec{x} \in F_{\beta(\gamma)}$ and $z = \pi^+_\gamma(I(F_{\beta(\gamma)})) \in \mathrm{range}(\pi^+_\gamma)$. **qed**

The closure properties of 4.1 will later be arranged by forming constructible hulls according to the following criterion:

4.2. Lemma. Let $X \subseteq F_\alpha$ and $\alpha \geq \omega$. Then the constructible hull $Z = \mathcal{F}(X \cup \{F_\alpha\})$ satisfies the property $Z = I[Z^{<\omega}]$.

Proof. Since$X \cup \{F_\alpha\} \subseteq F_{\alpha+1}$, $Z \subseteq F_{\alpha+1}$. We have to show that $Z \subseteq I[Z^{<\omega}]$. Any $z \in Z$ is generated from points in $X \cup \{F_\alpha\}$ by some composition of the operators I and S; in this composition the operation S will only yield elements of F_α. So if $z \in Z \setminus F_\alpha$ and $z \neq F_\alpha$ then it is of the form $z = I(F_\alpha, \varphi, \vec{y})$ where $\vec{y} \in Z$ and $z \in I[Z^{<\omega}]$. If $z = F_\alpha$ then $F_\alpha = I(F_\alpha, \dot{v}_0 \dot{=} \dot{v}_0) \in I[Z^{<\omega}]$. Finally if $z \in F_\alpha \cap Z$ then $z = I(F_\beta, \varphi, \vec{y})$ for some lexicographically minimal name $(F_\beta, \varphi, \vec{y})$. The components $F_\beta, \vec{y}$ of the minimal name can be obtained from z using the Skolem function $S(F_\alpha, \_, \_)$. Since Z is closed with respect to $S(F_\alpha, \_, \_)$ we get $F_\beta, \vec{y} \in Z$, hence $z \in I[Z^{<\omega}]$. **qed**

5. Directed Systems

Directed systems of fine levels will later be used to *extend* fine maps onto larger domains. We show here that wellfounded limits of such systems belong to the fine hierarchy:

5.1. Definition $(\mathcal{A}_i)_{i \in D}, (\pi_{ij})_{i \leq j \in D}$ is a *fine system* if it is a directed system of fine maps $\pi_{ij} \colon \mathcal{A}_i \to \mathcal{A}_j$ for $i \leq j$ from D. A fine system has a *direct limit* $\mathcal{A}, (\pi_i)_{i \in D}$ with fine maps $\pi_i \colon \mathcal{A}_i \to \mathcal{A}$, which is uniquely determined up to isomorphism.

5.2. Lemma Let $(\mathcal{A}_i)_{i \in D}, (\pi_{ij})_{i \leq j \in D}$ be a fine system of structures $\mathcal{A}_i = \mathcal{F}_{\alpha_i}$, with a direct limit $\mathcal{A} = (A, I^*, S^*, \in^*, <^*)$, $(\pi_i)_{i \in D}$. If $\in^*$ is strongly wellfounded then $\mathcal{A}$ is isomorphic to a fine level $\mathcal{F}_\alpha$.

Proof.

(1) $(A, \in^*)$ is extensional.

Proof. Let $a, b \in A$, $a \neq b$. Choose $i \in D$, $\bar{a}, \bar{b} \in F_{\alpha_i}$ such that $a = \pi_i(\bar{a})$, $b = \pi_i(\bar{b})$. Then $\bar{a} \neq \bar{b}$ and we can choose $\bar{c} \in F_{\alpha_i}$ such that $\bar{c} \in \bar{a} \leftrightarrow \bar{c} \notin \bar{b}$. Then $\pi_i(\bar{c}) \in^* a \leftrightarrow \pi_i(\bar{c}) \in^* b$. qed(1)

By the Mostowski isomorphism theorem, $(A, \in^*)$ is isomorphic to a transitive set, and we may conveniently assume that A is transitive and $\in^* = \in \restriction A$. The subsequent argument will be an induction on the following relation on $\{(i, F_\gamma) \mid i \in D,\ F_\gamma \in F_{\alpha_i}\}$:

$$(j, F_\delta) \tilde{<} (i, F_\gamma) \text{ iff } \pi_j(F_\delta) \in \pi_i(F_\gamma).$$

The wellfoundedness of $\in$ implies:

(2) $\quad \tilde{<}$ is a wellfounded relation.

We shall now state five properties about (i, F_γ), $i \in D$, $F_\gamma \in F_{\alpha_i}$ which will be proved by a simultaneous induction on $\tilde{<}$

(3) $\pi_i(F_\gamma) = F_\xi$ for some ordinal ξ.

(4) $I^* \upharpoonright F_\xi = I \upharpoonright F_\xi$, $<^* \upharpoonright F_\xi = < \upharpoonright F_\xi$, and $S^* \upharpoonright F_\xi = S \upharpoonright F_\xi$.

(5) $\forall \varphi \in \mathcal{L}_0 \forall \vec{y} \in F_\xi$: $I^*(F_\xi, \varphi, \vec{y}) = I(F_\xi, \varphi, \vec{y})$.

(6) $\forall x, y \in F_\xi$: $x <^*_{F_\xi} y$ iff $x <_{F_\xi} y$.

(7) $\forall \varphi \in \mathcal{L}_0 \forall \vec{y} \in F_\xi$: $S^*(F_\xi, \varphi, \vec{y}) = S(F_\xi, \varphi, \vec{y})$.

So fix (i, F_γ) and assume that (3) – (7) hold for all $\tilde{<}$-smaller pairs. First we prove (3) and (4) at (i, F_γ). We distinguish whether (i, F_γ) is a limit or a successor in $\tilde{<}$.

Case 1 ("limit"): For all $(j, F_\delta) \tilde{<} (i, F_\gamma)$ there is (k, F_η) such that $(j, F_\delta) \tilde{<} (k, F_\eta) \tilde{<} (i, F_\gamma)$.

Then let $F_\xi = \bigcup \{\pi_j(F_\delta) \mid (j, F_\delta) \tilde{<} (i, F_\gamma)\}$; note that the right hand side belongs to the fine hierarchy by the inductive asumption (3). Then:

(3) $\quad \pi_i(F_\gamma) = F_\xi$.

Proof. ($\subseteq$). Let $x \in \pi_i(F_\gamma)$. Choose $j \in D$ and $\bar{x} \in F_{\alpha_j}$ such that $x = \pi_j(\bar{x})$. Let $\bar{x} \in F_{\delta+1} \setminus F_\delta$. Take $l \in D$, $i, j \leq l$. Then $\pi_{jl}(\bar{x}) \in \pi_{il}(F_\gamma)$, $\pi_{jl}(F_\delta) \in \pi_{il}(F_\gamma)$, $\pi_j(F_\delta) \in \pi_i(F_\gamma)$, and thus $(j, F_\delta) \tilde{<} (i, F_\gamma)$. By the case assumption there is (k, F_η) such that $(j, F_\delta) \tilde{<} (k, F_\eta) \tilde{<} (i, F_\gamma)$. We may also assume that $j \leq k$. Then $\pi_{jk}(F_\delta) \in F_\eta$. The level F_η is closed under the I-function, hence $\pi_{jk}(\bar{x}) \in F_\eta$. So $x = \pi_k(\pi_{jk}(\bar{x})) \in \pi_k(F_\eta) \subseteq F_\xi$.

($\supseteq$). Conversely, let $x \in \pi_j(F_\delta)$, $(j, F_\delta) \tilde{<} (i, F_\gamma)$. Choose $k \in D$, $j, i \leq k$ and $\bar{x} \in F_{\alpha_k}$ such that $x = \pi_k(\bar{x})$. Then $\bar{x} \in \pi_{jk}(F_\delta) \in \pi_{ik}(F_\gamma)$ and $\bar{x} \in \pi_{ik}(F_\gamma)$ since $\pi_{ik}(F_\gamma)$ is transitive. Hence $x \in \pi_i(F_\gamma)$. qed(3, Case 1)

(4) is now clear since by the proof of (3) F_ξ is the union of structures on which, by induction assumption, I^* and I, $<^*$ and $<$, as well as N^* and N agree. Hence they agree on F_ξ.

Case 2 ("successor"): There is some $(j, F_\delta) \tilde{<} (i, F_\gamma)$ such that there is *no* (k, F_η) with
$(j, F_\delta) \tilde{<} (k, F_\eta) \tilde{<} (i, F_\gamma)$.
Then let $F_\rho = \pi_j(F_\delta)$ and set $\xi = \rho + 1$. We may assume that $j \geq i$.

(3) $\quad \pi_i(F_\gamma) = F_\xi$.

Proof. ($\subseteq$). Let $x \in \pi_i(F_\gamma)$. Choose $k \in D$, $k \geq j$ and $\bar{x} \in F_{\alpha_k}$ such that $x = \pi_k(\bar{x})$. Let $\bar{x} \in F_{\eta+1} \setminus F_\eta$. Since $x \in \pi_i(F_\gamma) \setminus \pi_k(F_\eta)$ we have $(k, F_\eta) \tilde{<} (i, F_\gamma)$. By the case assumption, $\pi_k(F_\eta) \subseteq \pi_j(F_\delta)$. Let $\bar{x} = I(F_\eta, \varphi, \vec{y})$.

$$\begin{aligned}\pi_k(\bar{x}) &= I^*(\pi_k(F_\eta), \varphi, \pi_k(\vec{y})), \quad \text{by the definition of } I^*;\\ &= I(\pi_k(F_\eta), \varphi, \pi_k(\vec{y})), \quad \text{by the inductive hypothesis (5);}\\ &\in F_{\rho+1} = F_\xi,\end{aligned}$$

since $\pi_k(F_\eta) \subseteq \pi_j(F_\delta) = F_\rho$.

($\supseteq$). Conversely, let $x \in F_\xi$. Let $x = I(F_\rho, \varphi, \vec{y})$, $\varphi \in \mathcal{L}_0$, $\vec{y} \in F_\rho$. Choose some $k \in D$, $k \geq j$, $\bar{x}, \vec{\bar{y}} \in F_{\alpha_k}$ such that $x = \pi_k(\bar{x})$, $\vec{y} = \pi_k(\vec{\bar{y}})$. By property (5) for (j, F_δ): $\bar{x} = I(\pi_{jk}(F_\delta), \varphi, \vec{\bar{y}})$. $\pi_{jk}(F_\delta) \in \pi_{ik}(F_\gamma)$, hence $\bar{x} \in \pi_{ik}(F_\gamma)$ and $x \in \pi_i(F_\gamma)$. qed(3, Case 2)

(4) follows from (4) – (7) at (j, F_δ).

In the proofs of (5) – (7) both cases will be treated together:

Proof of (5) at (i, F_γ). Let $\varphi(\dot{v}_0, \ldots, \dot{v}_n) \in \mathcal{L}_0$, $y_0, \ldots, y_{n-1} \in F_\xi$. Consider $j \in D$, $\bar{x} \in F_{\alpha_j}$. It suffices to see that

$$\pi_j(\bar{x}) \in I^*(F_\xi, \varphi, \vec{y}) \text{ iff } \pi_j(\bar{x}) \in I(F_\xi, \varphi, \vec{y}).$$

We may assume that j is sufficiently large so that there are $\bar{y}_0, \ldots, \bar{y}_{n-1} \in F_{\alpha_j}$ such that $y_0 = \pi_j(\bar{y}_0)$, $\ldots$, $y_{n-1} = \pi_j(\bar{y}_{n-1})$. Then:

$$\begin{aligned}\pi_j(\bar{x}) \in I^*(F_\xi, \varphi, \vec{y}) &\iff \bar{x} \in I(F_\gamma, \varphi, \vec{\bar{y}})\\ &\iff (F_\gamma, I, S, \in, <) \models \varphi[\vec{\bar{y}}, \bar{x}]\\ &\iff (F_\xi, I, S, \in, <) \models \varphi[\vec{y}, \pi_j(\bar{x})], \text{ by property (4);}\\ &\iff \pi_j(\bar{x}) \in I(F_\xi, \varphi, \vec{y})\end{aligned}$$

qed(5)

Proof of (6) at (i, F_γ). Let $x, y \in F_\xi$. Let $(F_\gamma, \varphi, \vec{c})$, $(F_\delta, \psi, \vec{d})$ be lexicographically minimal names for x, y respectively. Choose $i \in D$ and $\bar{x}, \bar{y}, F_{\bar{\gamma}}, \vec{\bar{c}}, F_{\bar{\delta}}, \vec{\bar{d}} \in F_{\alpha_i}$ such that $x, y, F_\gamma, \vec{c}, F_\delta, \vec{d} = \pi_i(\bar{x}, \bar{y}, F_{\bar{\gamma}}, \vec{\bar{c}}, F_{\bar{\delta}}, \vec{\bar{d}})$. By (4), π_i preserves I and $<$ and so $(F_{\bar{\gamma}}, \varphi, \vec{\bar{c}})$ is the lexicographically least name for $\bar{x}$. Similarly $(F_{\bar{\delta}}, \psi, \vec{\bar{d}})$ is the lexicographically least name for $\bar{y}$. Then:
$x <_{F_\xi} y$

iff $(F_\gamma, \varphi, \vec{c})$ is lexicographically less than $(F_\delta, \psi, \vec{d})$

iff $(F_{\bar{\gamma}}, \varphi, \vec{\bar{c}})$ is lexicographically less than $(F_{\bar{\delta}}, \psi, \vec{\bar{d}})$

iff $\quad \bar{x} <_{F_{\alpha_i}} \bar{y}$
iff $\quad x <^*_{F_\xi} y$. qed(6)

Proof of (7) at (i, F_γ). Let $\varphi(\dot{v}_0, \ldots, \dot{v}_n) \in \mathcal{L}_0$, $y_0, \ldots, y_{m-1} \in F_\xi$, $m \leq n$.

Case 1: $S(F_\xi, \varphi, \vec{y})$ is defined to be 0 by some default case of definition 2.2. Choose some $j \in D$, $j \geq i$, $\bar{y}_0, \ldots, \bar{y}_{m-1} \in F_{\alpha_j}$, $y_0 = \pi_j(\bar{y}_0)$, $\ldots$, $y_{m-1} = \pi_j(\bar{y}_{m-1})$. The map $\pi_j \colon (\pi_{ij}(F_\gamma), I, S, \in, <) \to (F_\xi, I, S, \in, <)$ is fine, and therefore $S(\pi_{ij}(F_\gamma), \varphi, \vec{\bar{y}})$ is also defined to be 0 by a corresponding default case of definition 2.2. Then $S^*(F_\xi, \varphi, \vec{y}) = \pi_j(S(\pi_{ij}(F_\gamma), \varphi, \vec{\bar{y}})) = \pi_j(0) = 0$, as required.

Case 2: The definition of $S(F_\xi, \varphi, \vec{y})$ does not fall under a default case. So $S(F_\xi, \varphi, \vec{y})$ is the $<_{F_\xi}$-minimal z for which there are $z_{m+1}, \ldots, z_n \in F_\xi$ satisfying

$$\mathcal{F}_\xi \models \varphi[\vec{y}, z, z_{m+1}, \ldots, z_n].$$

Choose some $j \in D$, $j \geq i$, $\bar{y}_0, \ldots, \bar{y}_{m-1}, \bar{z}, \bar{z}_{m+1}, \ldots, \bar{z}_n \in F_{\alpha_j}$ such that:

$$y_0 = \pi_j(\bar{y}_0),\ \ldots,\ y_{m-1} = \pi_j(\bar{y}_{m-1}),\ z = \pi_j(\bar{z}),\ z_{m+1} = \pi_j(\bar{z}_{m+1}),\ \ldots,\ z_n = \pi_j(\bar{z}_n).$$

The map $\pi_j \colon (\pi_{ij}(F_\gamma), I, S, \in, <) \to (F_\xi, I, S, \in, <)$ is fine, and therefore

$$\bar{z} = S(\pi_{ij}(F_\gamma), \varphi, \vec{\bar{y}}).$$

Then

$$\begin{aligned} S^*(F_\xi, \varphi, \vec{y}) &= \pi_j(S(\pi_{ij}(F_\gamma), \varphi, \vec{\bar{y}})) \\ &= \pi_j(\bar{z}) = z \\ &= S(F_\xi, \varphi, \vec{y}), \end{aligned}$$

as required. qed(7)

We can now prove the lemma:

Let $F_\theta = \bigcup\{\pi_j(F_\delta) \mid j \in D,\ F_\delta \in F_{\alpha_j}\}$. If $F_\theta = A$, we are done. Suppose $A \setminus F_\theta \neq \emptyset$. Take $x \in A \setminus F_\theta$. Choose $i \in D$, $\bar{x} \in F_{\alpha_i}$ such that $x = \pi_i(\bar{x})$. Let $\bar{x} \in F_{\gamma+1} \setminus F_\gamma$, $F_\gamma \in F_{\alpha_i}$. $\bar{x} = I(F_\gamma, \varphi, \vec{y})$ for some $\varphi \in \mathcal{L}_0$, $\vec{y} \in F_\gamma$. By (5), $x = I(\pi_i(F_\gamma), \varphi, \pi_i(\vec{y}))$. Clearly $\pi_i(F_\gamma) \subseteq F_\theta$. If $\pi_i(F_\gamma) \in F_\theta$, then $x \in F_\theta$, contradicting the choice of x. Hence $\pi_i(F_\gamma) = F_\theta$.

The preceding argument shows that $A \subseteq F_{\theta+1}$. Conversely, $F_{\theta+1} \subseteq A$, since $F_\theta = \pi_i(F_\gamma) \in A$ and A is closed under the function $I(F_\theta, \_, \_)$. Hence

$A = F_{\theta+1}$ where $F_\theta = \pi_i(F_\gamma)$ and then (5), (6), and (7) yield $I^* = I \upharpoonright F_{\theta+1}$, $<^* = < \upharpoonright F_{\theta+1}$, and $S^* = S \upharpoonright F_{\theta+1}$. **qed**

6. Extensions

Large Cardinal Theory can be viewed as the theory of elementary embeddings of transitive models of set theory. Large cardinal strength corresponds to the richness of the collection of such embeddings. Relevant techniques include the formation of ultrapowers by ultrafilters or the construction of extensions by extenders. Since an extender can be defined to be an elementary map itself, one is producing new elementary maps out of given ones. In this chapter we shall construct extensions of fine levels which are related to the proof of the Jensen Covering Theorem for L. We shall first consider a (trivial) presentation of L as the direct limit of a directed system $\mathcal{S}$ of "small" structures. In the construction of extensions we shall employ subsystems of $\mathcal{S}$.

6.1. Definition. Define a directed partial order $(D, \leq)$:
$D = \{(\mu, p) \mid \mu \in \mathrm{On},\ p \subseteq \{F_\xi \mid \xi \in \mathrm{On}\},\ p \text{ is finite}\}$ and $(\mu, p) \leq (\nu, q)$ iff $\mu \leq \nu$ and $p \subseteq q$.

For $(\mu, p) \in D$ let $\pi_{\mu p} \colon \mathcal{F}_{\alpha(\mu,p)} \tilde{\leftrightarrow} \mathcal{F}(F_\mu \cup p)$ be the uniquely defined isomorphism given by the condensation lemma 4.1; note that the closure conditions of 4.1 are satisfied by 4.2. For $(\mu, p) \leq (\nu, q)$, we have $\mathcal{F}(F_\mu \cup p) \subseteq \mathcal{F}(F_\nu \cup q)$ and we can define the fine map

$$\pi_{\mu p, \nu q} = \pi_{\nu q}^{-1} \circ \pi_{\mu p} \colon \mathcal{F}_{\alpha(\mu,p)} \to \mathcal{F}_{\alpha(\nu,q)}.$$

The system

$$\mathcal{S} = (\mathcal{F}_{\alpha(\mu,p)})_{(\mu,p) \in D}, (\pi_{\mu p, \nu q})_{(\mu,p) \leq (\nu,q)}$$

is fine and $(L, I, S, \in, <), (\pi_{\mu p})_{(\mu,p) \in D}$ is the transitive direct limit of the system $\mathcal{S}$.

6.2. Definition. Let $\mathcal{F}_\alpha, \mathcal{F}_\delta$ be fine levels, $\alpha < \delta \leq \infty$. We say that $\mathcal{F}_\alpha$ is a *base* for $\mathcal{F}_\delta$ if for every $\mu < \alpha$ and every finite $p \subseteq \{F_\xi \mid \xi < \delta\}$ we have $\alpha(\mu, p) < \alpha$.

We shall extend a fine map E from $\mathcal{F}_\alpha$ to a larger domain $\mathcal{F}_\delta$: Fix a fine map $E \colon \mathcal{F}_\alpha \to \mathcal{F}_\beta$ where α is a limit ordinal greater than ω and where the range of E is *cofinal* in $\mathcal{F}_\beta$, i.e., $\forall y \in F_\beta\ \exists x \in F_\alpha\ y \in E(x)$. Cofinality can always be arranged by taking β minimal such that E maps into $\mathcal{F}_\beta$. Fix a fine level $\mathcal{F}_\delta$ such that $\mathcal{F}_\alpha$ is a base for $\mathcal{F}_\delta$.

The subsequent construction defines the *extension* of $\mathcal{F}_\delta$ by E.

Define a subsystem $\mathcal{S}_0$ of $\mathcal{S}$ by: $D_0 = \{(\mu,p)\,|\,\mu < \alpha,\ p \subseteq \{F_\xi\,|\,\xi < \delta\},\ p \text{ is finite}\}$ and

$$\mathcal{S}_0 = (\mathcal{F}_{\alpha(\mu,p)})_{(\mu,p)\in D_0}, (\pi_{\mu p,\nu q})_{(\mu,p)\leq(\nu,q)\in D_0}.$$

By Proposition 2.6,

$$F_\delta = \bigcup\{\mathcal{F}(F_\mu \cup p)\,|\,(\mu,p)\in D_0\}$$

and so

$$\mathcal{F}_\delta, (\pi_{\mu p})_{(\mu,p)\in D_0}$$

is the transitive direct limit of the system $\mathcal{S}_0$.

The base property 6.2 implies immediately:

(1) If $(\mu,p)\in D_0$ then $\mathcal{F}_{\alpha(\mu,p)} \in \mathcal{F}_\alpha$.

(2) If $(\mu,p)\leq(\nu,q)\in D_0$ then $\pi_{\mu p,\nu q}\in\mathcal{F}_\alpha$.

Proof. Set $\pi = \pi_{\mu p,\nu q}$. Let $p = \{p_0,\ldots,p_{m-1}\}$ and $p'_i = \pi^{-1}_{\mu p}(p_i)$, $p''_i = \pi^{-1}_{\nu q}(p_i)$ for $i < m$. For arbitrary x ,y we have $\pi(x) = y$ iff $x \in F_{\alpha(\mu,p)}$, $y \in F_{\alpha(\nu,q)}$ and there is a term t of the language $\mathcal{L}$ and a tuple $\vec{z}\in F_\mu$ such that

$$x = t(p'_0,\ldots,p'_{m-1},\vec{z}) \text{ and } y = t(p''_0,\ldots,p''_{m-1},\vec{z}).$$

The latter property can be expressed by an $\mathcal{L}$-formula of the form:

$$\begin{aligned}\psi \equiv\ &\dot v_{2m} \doteq t(\dot v_0,\ldots,\dot v_{m-1},\dot v_{2m+4},\ldots,\dot v_n)\\ &\dot\wedge\, \dot v_{2m+1} \doteq t(\dot v_m,\ldots,\dot v_{2m},\dot v_{2m+4},\ldots,\dot v_n)\\ &\dot\wedge\, \dot v_{2m+4}\dot\in\dot v_{2m+2}\dot\wedge\ldots\dot\wedge\dot v_n\dot\in\dot v_{2m+2}\\ &\dot\wedge\, \dot\neg\dot v_{2m+3} \doteq 0.\end{aligned}$$

The last conjunct is not relevant for the meaning of the formula but is put in for later use by the S-operation. For a fixed term t the condition on $\pi(x) = y$ becomes:

$$\exists\dot v_{2m+3}\ldots\dot v_n\ \psi[p'_0,\ldots,p'_{m-1},p''_0,\ldots,p''_{m-1},x,y,F_\mu]$$

Using the S-operation this is equivalent to

$$S(F_\tau, \psi, p'_0, \ldots, p'_{m-1}, p''_0, \ldots, p''_{m-1}, x, y, F_\mu) \neq 0,$$

where $\tau = \alpha(\nu, q)$ is big enough so that all parameters and all existential witnesses are contained in F_τ.

The set Ψ of all $\mathcal{L}$-formulas ψ as above is definable over $(F_\omega, \in)$ and is thus an element of $F_{\omega+\omega}$. By quantifying over all $\psi \in \Psi$ we are now able to define the map π :

$$\begin{aligned} \pi(x) = y &\iff \exists \psi \in \Psi\, S(F_\tau, \psi, p'_0, \ldots, p'_{m-1}, p''_0, \ldots, p''_{m-1}, x, y, F_\mu) \neq 0 \\ &\iff S(F_{\tau+1}, \chi, p'_0, \ldots, p'_{m-1}, p''_0, \ldots, p''_{m-1}, x, y, F_\mu, F_\tau) \neq 0, \end{aligned}$$

where χ is the formula :

$$\dot{\neg}\dot{S}(\dot{v}_{2m+3}, \dot{v}_{2m+4}, \dot{v}_0, \ldots, \dot{v}_{m-1}, \dot{v}_m, \ldots, \dot{v}_{2m-1}, \dot{v}_{2m}, \dot{v}_{2m+1}, \dot{v}_{2m+2}) \dot{=} 0.$$

So we can define the map π by an $\mathcal{L}_0$-formula. Using Kuratowski pairing and projection functions inside $\mathcal{F}_\alpha$ we see that π is an element of the base $\mathcal{F}_\alpha$. qed(2)

By (1) and (2), the system $\mathcal{S}_0$ can be mapped componentwise by the map E. For $(\mu, p) \in D_0$ let $\mathcal{F}_{\alpha^*(\mu,p)} = (\mathcal{F}_{\alpha(\mu,p)})$; for $(\mu, p) \leq (\nu, q) \in D_0$ let $\pi^*_{\mu p,\nu q} = E(\pi_{\mu p,\nu q})$.

(3) $\mathcal{S}^*_0 = (\mathcal{F}_{\alpha^*(\mu,p)})_{(\mu,p)\in D_0}, (\pi^*_{\mu p,rq})_{(\mu,p)\leq(r,q)\in D_0}$ is a fine system.

Proof. The fact that $\pi_{\mu p,\nu q} \colon \mathcal{F}_{\alpha(\mu,p)} \to \mathcal{F}_{\alpha(\nu,p)}$ is fine can be expressed by a schema of universal formulas where the universal quantifiers may be restricted to $F_{\alpha(\nu,p)}$. Using the S-operation this can be expressed quantifier-free in $\mathcal{F}_\alpha$. The fine map $E \colon \mathcal{F}_\alpha \to \mathcal{F}_\beta$ lifts the schema up to $\mathcal{F}_\beta$, hence $\pi^*_{\mu p,\nu q} \colon \mathcal{F}_{\alpha^*(\mu,p)} \to \mathcal{F}_{\alpha^*(\nu,q)}$ is fine.

The commutativity of the fine system lifts up by the same method. qed(3)

Let $\mathcal{A}, (\pi^*_{\mu p})_{(\mu,p)\in D_0}$ be a direct limit of the system $\mathcal{S}^*$, $\mathcal{A} = (A, I^*, S^*, \in^*, <^*)$. We show that $\mathcal{F}_\beta$ is isomorphic to an initial segment of $\mathcal{A}$. Define a map $\sigma \colon F_\beta \to A$ by: for $y \in F_\beta$ take $(\mu, 0) \in D_0$ such that $y \in E(F_\mu)$; then let $\sigma(y) = \pi^*_{\mu 0}(y)$.

(4) The definition of $\sigma(y)$ is independent of the choice of μ.

Proof. Let $y \in E(F_\mu)$ and $y \in E(F_\nu)$ with $\mu \leq r$. $\pi_{\mu 0,\nu 0} = \mathrm{id} \restriction F_\mu$, and this is preserved by the lifting by E: $\pi^*_{\mu 0,\nu 0} = \mathrm{id} \restriction E(F_\mu)$. Hence $\pi^*_{\mu 0}(y) = \pi^*_{\nu 0} \circ \pi^*_{\mu 0,\nu 0}(y) = \pi^*_{\nu 0}(y)$. qed(4)

(5) $\sigma \colon \mathcal{F}_\beta \to \mathcal{A}$ is fine.

Proof. Since σ is the union of the fine maps $\pi^*_{\mu 0} \colon \mathcal{F}_{\alpha^*(\mu,0)} \to \mathcal{A}$ qed(5)

(6) $\sigma'' F_\beta$ is an $\in^*$-initial segment of $\mathcal{A}$.

Proof. Let $z \in^* \pi^*_{\mu 0}(y)$. Choose $(\nu, q) \in D_0$, $(\mu, 0) \le (\nu, q)$ and $\bar{z} \in F_{\alpha^+(\nu,q)}$ such that $z = \pi^*_{\nu q}(\bar{z})$. Then $\bar{z} \in \pi^*_{\mu 0,\nu q}(y)$. Since $\pi_{\mu 0,\nu q}$ is the identity on F_μ, $\pi^*_{\mu 0,\nu q}$ is the identity on $F_{\alpha^*(\mu,0)}$. Hence $\bar{z} \in y$. Also $z = \pi^*_{\nu q}(\bar{z}) = \pi^*_{\nu q} \circ \pi^*_{\mu 0,\nu q}(\bar{z}) = \pi^*_{\mu 0}(\bar{z}) \in \sigma'' F_\beta$. qed(6)

By (5) and (6) we may assume $\mathcal{F}_\beta$ is an initial segment of $\mathcal{A}$.

Since the fine systems are nicely connected by the map E, one can define a fine map from the limit of $\mathcal{S}_0$ into the limit of $\mathcal{S}_0^*$: for $x \in F_\delta$, $x = \pi_{\mu p}(\bar{x})$ define $\pi_E(x) = \pi^*_{\mu p}(E(\bar{x}))$.

(7) The definition of $\pi_E(x)$ is independent of the choice of $(\mu, p) \in D_0$ and $\bar{x}$.

Proof. Let $x = \pi_{\mu p}(\bar{x}) = \pi_{\nu q}(x')$. Choose $(\lambda, r) \in D_0$ such that $(\mu, p) \le (\lambda, r)$ and $(\nu, q) \le (\lambda, r)$. Then $\pi_{\mu p,\lambda r}(\bar{x}) = \pi_{\nu q,\lambda r}(x')$ and $\pi^*_{\mu p,\lambda r}(E(\bar{x})) = \pi^*_{\nu q,\lambda r}(E(x'))$. So

$$\begin{aligned}
\pi^*_{\mu p}(E(\bar{x})) &= \pi^*_{\lambda r} \circ \pi^*_{\mu p,\lambda r}(E(\bar{x})) \\
&= \pi^*_{\lambda r} \circ \pi^*_{\nu q,\lambda r}(E(x')) \\
&= \pi^*_{\nu q}(E(x')).
\end{aligned}$$

qed(7)

(8) $\pi_E \colon \mathcal{F}_\delta \to \mathcal{A}$ is fine.

Proof. Because

$$\begin{tikzcd}
\mathcal{F}_{\alpha^*(\mu,p)} \arrow[r, "\pi^*_{\mu p}"] & \mathcal{A} \\
\mathcal{F}_{\alpha(\mu,p)} \arrow[u, "E \restriction F_{\alpha(\mu,p)}"] \arrow[r, "\pi_{\mu p}"] & \mathcal{F}_\delta \arrow[u, "\pi_E"']
\end{tikzcd}$$

is a commutative diagram of fine maps. qed(8)

(9) π_E extends the map E.

Proof. Let $x \in F_\alpha$. Choose $(\mu, 0) \in D_0$ such that $x \in F_\mu$. $F_{\alpha(\mu,0)} = F_\mu$ and $\pi_{\mu 0} = \mathrm{id} \restriction F_\mu$. Then $\pi^*_{\mu 0} = \mathrm{id} \restriction E(F_\mu)$ and $\pi_E(x) = \pi_E(\pi_{\mu 0}(x)) = \pi^*_{\mu 0}(E(x)) = E(x)$. qed(9)

6.3. Definition. We call the structure $\mathcal{A} = (A, I^*, S^*, \in^*, <^*)$ defined above the *extension* of $\mathcal{F}_\delta$ by E; we write $\mathcal{A} = \mathrm{Ext}(\mathcal{F}_\delta, E)$. In case $(A, \in^*)$ is strongly wellfounded then $\mathcal{A}$ will be identified with an isomorphic fine level $\mathcal{F}_\eta$ (see lemma 5.2). The map $\pi_E \colon \mathcal{F}_\delta \to \mathcal{A}$ is called the *extension map* of $\mathcal{F}_\delta$ by E.

In the strongly wellfounded case, the map π_E will be better than fine:

6.4. Lemma. Let $\pi_E \colon \mathcal{F}_\delta \to \mathcal{F}_\eta$ be the extension map of $\mathcal{F}_\delta$ by E. Then π_E is fine up to $\mathcal{F}_\eta$.

Proof. We have to show that $\pi_E'' F_\delta$ is constructibly closed up to $\mathcal{F}_\eta$, i.e., that properties 3.3(a) and (b) are satisfied.

(a) Consider $\varphi \in \mathcal{S}_0$, $\vec{x} \in F_\delta$ and assume that $z = I(F_\eta, \varphi, \pi_E(\vec{x})) \in F_\eta$. We have to show that $z \in \pi_E'' F_\delta$.
Take $(\mu, p) \in D_0$ such that $\vec{x} \in \mathrm{range}(\pi_{\mu p})$ and $z \in \mathrm{range}(\pi^*_{\mu p})$. Let $\vec{x} = \pi_{\mu p}(\vec{u})$, $z = \pi^*_{\mu p}(y)$. Then $\pi_E(\vec{x}) = \pi^*_{\mu p}(E(\vec{u})) \in \mathrm{range}(\pi^*_{\mu p})$. By the definition of I,

$$\mathcal{F}_\eta \models \forall v \, (v \in z \leftrightarrow \varphi(\pi_E(\vec{x}), v)).$$

The map $\pi^*_{\mu p} \colon \mathcal{F}_{\alpha^*(\mu p)} \to \mathcal{F}_\eta$ is fine and preserves such universal statements downwards:

$$\mathcal{F}_{\alpha^*(\mu,p)} \models \forall v \, (v \in y \leftrightarrow \varphi(E(\vec{u}), v)).$$

The set y is uniquely determined by this property. The map

$$E \restriction F_{\alpha(\mu,p)} \colon \mathcal{F}_{\alpha(\mu,p)} \to \mathcal{F}_{\alpha^*(\mu,p)}$$

is sufficiently elementary so that the definable point y has to be in the range of $E \restriction F_{\alpha(\mu,p)}$. Take $\bar{y} \in F_{\alpha(\mu,p)}$ such that $y = E(\bar{y})$. Then

$$z = \pi^*_{\mu p}(y) = \pi^*_{\mu p}(E(\bar{y})) = \pi_E(\pi_{\mu p}(\bar{y})) \in \mathrm{range}(\pi_E).$$

(b) Consider $\varphi(\dot{v}_0, \ldots, \dot{v}_n) \in \mathcal{S}_0$, $\vec{x} = x_0, \ldots, x_{m-1} \in F_\delta$, $m \leq n$ and assume that $S(F_\eta, \varphi, \pi_E(\vec{x})) \neq 0$, i.e., it is defined non-trivially. Then there are $y_{m+1}, \ldots, y_n \in F_\eta$ such that

$$\mathcal{F}_\eta \models \varphi[\pi_E(x_0), \ldots, \pi_E(x_{m-1}), z, y_{m+1}, \ldots, y_n],$$

and z is $<_{F_\eta}$-smallest with this property. By Proposition 2.6 we may assume that z is a level of the fine hierarchy. Take $(\mu, p) \in D_0$ such that $\vec{x} \in \mathrm{range}(\pi_{\mu p})$ and $z, y_{m+1}, \ldots, y_n \in \mathrm{range}(\pi^*_{\mu p})$. Let $\vec{x} = \pi_{\mu p}(\vec{u})$, $z = \pi^*_{\mu p}(w)$, $y_{m+1} = \pi^*_{\mu p}(w_{m+1})$, $\ldots$, $y_n = \pi^*_{\mu p}(w_n)$. By definition of S

$$\begin{aligned}\mathcal{F}_\eta \models \forall v_m \ldots v_n\, (v_m = I(v_m, v_0 = v_0) \wedge v_m \in z \\ \longrightarrow \neg\varphi(\pi_E(x_0), \ldots, \pi_E(x_{m-1}), v_m, v_{m+1}, \ldots, v_n)).\end{aligned}$$

Since $\pi^*_{\mu p}$ preserves universal $\mathcal{L}$-statements downwards:

$$\begin{aligned}&\mathcal{F}_{\alpha^*(\mu,p)} \models \varphi[E(u_0), \ldots, E(u_{m-1}), w, w_{m+1}, \ldots, w_m] \wedge w = I(w, v_0 \doteq v_0)\\ &\mathcal{F}_{\alpha^*(\mu,p)} \models \forall v_m \ldots v_n\, (v_m = I(v_m, v_0 \doteq v_0) \wedge v_m \in w \\ &\qquad\qquad \longrightarrow \neg\varphi(E(u_0), \ldots, E(u_{m-2}), v_m, v_{m+2}, \ldots, v_n)).\end{aligned}$$

The set w is uniquely determined by these properties and so is in $\mathrm{range}(E)$. Let $w = E(\bar{w})$. Then $z = \pi^*_{\mu p}(w) = \pi^*_{\mu p}(E(\bar{w})) = \pi_E(\pi_{\mu p}(\bar{w})) \in \mathrm{range}(\pi_E)$. **qed**

By this lemma and lemma 3.8, an extension $\pi_E \colon \mathcal{F}_\delta \to \mathcal{F}_\eta$ can be lifted to a fine map $\pi^*_E \colon \mathcal{F}_{\delta+1} \to \mathcal{F}_{\eta+1}$. In this way, the smallest $\mathcal{F}_{\delta+1}$ for which $\mathcal{F}_\alpha$ is not a base can still be mapped over, which will give a covering set in the proof of the covering theorem.

7. Strong Maps.

For the proof of the covering theorem for L we construct *strong maps* which induce wellfounded extensions.

7.1. Definition. Let $E \colon \mathcal{F}_\alpha \to \mathcal{F}_\beta$ be a cofinal map with limit ordinal α.

(a) The map E is *strong for* $\mathcal{F}_\delta$, if $\mathrm{Ext}(\mathcal{F}_\delta, E)$ is wellfounded in case $\mathcal{F}_\alpha$ is a base for $\mathcal{F}_\delta$.

(b) The map E is *strong for the fine hierarchy* ($\mathcal{F}$-strong) if E is strong for every $\mathcal{F}_\delta$.

In the construction of strong maps we use the following criterium for non-strength:

7.2. Lemma. Let $E \colon \mathcal{F}_\alpha \to \mathcal{F}_\beta$ be cofinal, α a limit ordinal. Assume that E is not strong for $\mathcal{F}_\delta$. Then there is a sequence $(\mu_0, p_0) \le (\mu_1, p_1) \le \cdots \in D$ with $\mu_n < \alpha$, $p_n \subseteq F_\delta$ and a sequence $y_0, y_1, \ldots \in F_\beta$ such that

$y_{n+1} \in E(\pi_{\mu_n p_n, \mu_{n+1} p_{n+1}})(y_n)$ for all $n < \omega$. A sequence $y_0, y_1, \ldots \in F_\beta$ like this is called a *vicious sequence* for E, $\mathcal{F}_\delta$.

Proof. Use the notations from the previous chapter. Let $\mathcal{A} = (A, I^*, S^*, \in^*, <^*)$ be the nonwellfounded extension. There is a sequence $y_0^*, y_1^*, \ldots \in A$ with $y_{n+1}^* \in^* y_n^*$ for all $n < \omega$. For $n < \omega$ choose an index $(\mu_n, p_n) \in D$, $\mu_n < \alpha$, $p_n \subseteq F_\delta$ and $y_n \in F_\beta$ $y_n^* = \pi_{\mu_n p_n}(y_n)$. We may assume that $(\mu_n, p_n) \leq (\mu_{n+1}, p_{n+1})$ for all $n < \omega$. Then

$$y_{n+1}^* = \pi^*_{\mu_{n+1} p_{n+1}}(y_{n+1}) \in^* y_n^* = \pi^*_{\mu_n p_n}(y_n)$$

implies

$$y_{n+1} \in (\pi^*_{\mu_{n+1} p_{n+1}})^{-1} \circ \pi^*_{\mu_n p_n}(y_n) = E(\pi_{\mu_n p_n, \mu_{n+1} p_{n+1}})(y_n)$$

as required. **qed**

The following proof is based on the idea that the counterexample $\mathcal{F}_\delta$ to the strength of E can not exist, if the vicious sequence $y_0, y_1, \ldots$ were taken from range(E); in that case the infinite descent could be pulled back to $\mathcal{F}_\alpha$ and to the $\in$-relation of $\mathcal{F}_\delta$, contradicting the wellfoundedness of $\in$.

7.3. Lemma. Let $\beta > \omega$ be a limit ordinal and let $X \subseteq \beta$ be cofinal in β with $\mu = \mathrm{card}(X)$ regular and uncountable. Then there is an $\mathcal{F}$-strong map $E \colon \mathcal{F}_\alpha \to \mathcal{F}_\beta$ such that $X \subseteq \mathrm{range}(E)$ and $\mathrm{card}(F_\alpha) = \mu$.

Proof. We construct a continuous tower $(Y_\xi \,|\, \xi < \mu)$ of substructures of $\mathcal{F}_\beta$ whose union will be range(E). Present X as $X = \bigcup\{X_\xi \,|\, \xi < \mu\}$, where $\mathrm{card}(X_\xi) < \mu$ for $\xi < \mu$. By simultaneous induction on $\xi < \mu$ define a sequence $(E_\xi \,|\, \xi \leq \mu)$ of fine maps $E_\xi \colon \mathcal{F}_{\alpha_\xi} \to \mathcal{F}_\beta$, $Y_\xi = \mathrm{range}(E_\xi)$ such that

(1) $\xi \leq \zeta \leq \mu \longrightarrow Y_\xi \subseteq Y_\zeta$.
(2) $\zeta \leq \mu$, limit $\zeta \longrightarrow Y_\zeta = \bigcup_{\xi<\zeta} Y_\xi$.
(3) $X_\xi \subseteq Y_{\xi+1}$, $\mathrm{card}(Y_\xi) < \mu$.
(4) If $\xi < \mu$ and E_ξ is not $\mathcal{F}$-strong take $\mathcal{F}_\delta$ with δ minimal such that E_ξ is not strong for $\mathcal{F}_\delta$. Then let $Y_{\xi+1}$ contain a vicious sequence for E_ξ, $\mathcal{F}_\delta$ as a subset.

The construction is possible since we basically have to include a set of cardinality $< \mu$ in going from Y_ξ to $Y_{\xi+1}$ and form a constructible closure.

Then set $E = E_\mu$, $\alpha = \alpha_\mu$. Obviously $X \subseteq \mathrm{range}(E) = Y_\mu$ and $\mathrm{card}(F_\alpha) = \mathrm{card}(Y_\mu) = \mu$. We show that E is $\mathcal{F}$-strong. Assume for a contradiction that $\mathcal{F}_\alpha$ is a base for $\mathcal{F}_\eta$ but $\mathrm{Ext}(\mathcal{F}_\eta, E)$ is not wellfounded. By 7.2 there is a sequence $(\mu_0, p_0) \leq (\mu_1, p_1) \leq \cdots \in D$ with $\mu_n < \alpha$, $p_n \subseteq F_\delta$ and a vicious sequence $z_0, z_1, \ldots \in F_\beta$ such that

$$z_{n+1} \in E(\pi_{\mu_n p_n, \mu_{n+1} p_{n+1}})(z_n)$$

for $n < \omega$. The structure $\mathcal{F}_\alpha$ is the union of the continuous tower $(\tilde{Y}_\xi)_{\xi<\mu}$, where $\tilde{Y}_\xi = E^{-1}{}''Y_\xi$. Let H be a transitive structure containing all the sets mentioned so far and reflecting enough properties of V for the following argument. A straightforward Löwenheim-Skolem argument yields an elementary substructure W of H such that

(5) for $n < \omega$: $\mu_n \in W$, $p_n \in W$.

(6) $F_\alpha \in W$, $F_\delta \in W$.

(7) $W \cap F_\alpha = \tilde{Y}_\xi$ for some $\xi < \mu$.

Let $\sigma \colon \tilde{H} \cong W \prec H$, $\tilde{H}$ transitive be the Mostowski transitivisation. By (7), $\sigma^{-1}(F_\alpha)$ is the transitivisation of $\tilde{Y}_\xi \cong Y_\xi$, hence

(8) $\quad \sigma^{-1}(F_\alpha) = F_{\alpha_\xi}$.

Also, for that reason:

(9) $\quad E_\xi = E \circ \sigma \restriction F_{\alpha_\xi}$.

By the absoluteness of the relevant notions and elementarity

(10) $\quad \mathcal{F}_{\alpha_\xi}$ is a base for $\mathcal{F}_{\tilde{\delta}}$, where $\tilde{\delta} = \sigma^{-1}(\delta)$.

(11) $\quad E_\xi$ is not $\mathcal{F}$-strong.

Proof. For $n < \omega$ let

$$\tilde{\mu}_n = \sigma^{-1}(\mu_n),\ \tilde{p}_n = \sigma^{-1}(p_n),\ \tilde{\pi}_{n,n+1} = \sigma^{-1}(\pi_{\mu_n p_n, \mu_{n+1} p_{n+1}}) \in F_{\alpha_\xi}.$$

Then by elementarity, $\pi_{\tilde{\mu}_n \tilde{p}_n, \tilde{\mu}_{n+1} \tilde{p}_{n+1}} = \tilde{\pi}_{n,n+1}$, and for $n < \omega$:

$$\begin{aligned} z_{n+1} &\in E(\pi_{\mu_n p_n, \mu_{n+1} p_{n+1}})(z_n) \\ &= E \circ \sigma(\pi_{\tilde{\mu}_n \tilde{p}_n, \tilde{\mu}_{n+1} \tilde{p}_{n+1}})(z_n) \\ &= E_\xi(\pi_{\tilde{\mu}_n \tilde{p}_n, \tilde{\mu}_{n+1} \tilde{p}_{n+1}})(z_n). \end{aligned}$$

qed(11)

By the initial construction, let δ be minimal such that E_ξ is not strong for $\mathcal{F}_\delta$. By (4) above there are indices (ν_n, q_n) for $n < \omega$, such that $\nu_n < \alpha$ and $q_n \subseteq F_\delta$ and a vicious sequence $y_0, y_1, \ldots \in Y_{\xi+1}$ such that

$$y_{n+1} \in E_\xi(\pi_{\nu_n q_n, \nu_{n+1} q_{n+1}})(y_n)$$

for $n < \omega$. We can apply E^{-1} to both sides of the relation:

$$\begin{aligned} E^{-1}(y_{n+1}) &\in E^{-1} \circ E_\xi(\pi_{\nu_n q_n, \nu_{n+1} q_{n+1}})(E^{-1}(y_n)) \\ &= \sigma(\pi_{\nu_n q_n, \nu_{n+1} q_{n+1}})(E^{-1}(y_n)) \\ &= \pi_{\sigma(\nu_n)\sigma(q_n), \sigma(\nu_{n+1})\sigma(q_{n+1})})(E^{-1}(y_n)) \end{aligned}$$

Finally apply $\pi_{\sigma(\nu_{n+1})\sigma(p_{n+1})}$ to both sides:

$$\pi_{\sigma(\nu_{n+1})\sigma(p_{n+1})}(E^{-1}(y_{n+1})) \in \pi_{\sigma(\nu_n)\sigma(p_n)}(E^{-1}(y_n)),$$

which is a descending $\in$-chain in F_η. Contradiction. **qed**

8. The Jensen Covering Theoren for L.

8.1. Theorem. Assume, $0^\sharp$ does not exist. Then L *covers* V, which means: $\forall X \subseteq \mathrm{On}\, \exists Y \subseteq \mathrm{On}\, (X \subseteq Y \wedge Y \in L \wedge \mathrm{card}(Y) \leq \mathrm{card}(X) + \aleph_1)$.

Proof. Assume L does not cover V. Let $X \subseteq \beta$ be a counterexample to covering with β choosen minimally. We can assume that $\aleph_1 \subseteq X$ and that $\mu = \mathrm{card}(X)$ is choosen minimally with the stated properties.

(1) There is no $Y \supseteq X$, $Y \in L$, such that $\mathrm{card}^L(Y) < \beta$.

Proof. Assume to the contrary that $Y \supseteq X$, $Y \in L$, and there is $f \in L$, $\bar\beta < \beta$, $f\colon \bar\beta \leftrightarrow Y$. Let $\bar X = f^{-1\prime\prime}X$. By the minimality of β there is $\bar Y \in L$, $\bar X \subseteq \bar Y \subseteq \bar\beta$ with $\mathrm{card}(\bar Y) \leq \mathrm{card}(\bar X) + \aleph_1$. Then $f''\bar Y \in L$ is a "covering set" for X, which contradicts our assumptions on X. qed(1)

(2) β is a cardinal in L;

this follows immediately from (1).

(3) X is cofinal in β.

(4) $\mathrm{card}(X) \geq \aleph_1$,

since $\aleph_1 \subseteq X$.

(5) $\mathrm{card}(X)^+ < \beta$.

Proof. If $\beta < \mathrm{card}(X)^+$, we could take $\beta \in L$ to cover X. If $\beta = \mathrm{card}(X)^+$, X could not be cofinal in β. qed(5)

(6) $\mathrm{card}(X)$ is regular.

Proof. Assume that μ is singular. Let $\lambda = \mathrm{cof}(\mu) < \mu$. Write $X = \bigcup_{\xi<\lambda} X_\xi$ where each $\mathrm{card}(X_\xi) < \mu$ and $\sup(X_\xi) < \beta$. For each $\xi < \lambda$ choose $Y_\xi \in L$, $X_\xi \subseteq Y_\xi$, $\mathrm{card}(Y_\xi) \leq \mathrm{card}(X)$ using the minimality of β. We can assume that $Y_\xi \subseteq \mathrm{On}$ and $\sup(Y_\xi) < \beta$. Set $\iota = \mathrm{card}(X)^+$. There is a map $h\colon \beta \leftrightarrow F_\beta$ since β is an L-cardinal. Let $Z = \{h^{-1}(Y_\xi) \,|\, \xi < \lambda\} \subseteq \beta$; $\mathrm{card}(Z) \leq \lambda < \mu$, and by the minimality of μ there is $W \in L$, $Z \subseteq W$, $\mathrm{card}(W) \leq \mu$. Set

$$Y = \bigcup\{h(\eta) \,|\, \eta \in W,\, h(\eta) \subseteq \mathrm{On},\, \mathrm{otp}(h(\eta)) < \iota\}.$$

$Y \supseteq \bigcup\{h(\eta) \,|\, \eta \in Z\} = \bigcup_{\xi<\lambda} Y_\xi \supseteq \bigcup_{\xi<\lambda} X_\xi = X$, $Y \in L$, since the definition takes part inside of L, $\mathrm{card}^L(Y) \leq \iota$, since $\mathrm{card}^L(W) \leq \iota$ and $\mathrm{card}^L(h(\eta)) < \iota$ for every $h(\eta)$ used in the definition of Y. But this is a contradiction to (1). qed(6)

We are now in a position to use Lemma 7.3: There is an $\mathcal{F}$-strong map $E\colon \mathcal{F}_\alpha \to \mathcal{F}_\beta$ such that $X \subseteq \mathrm{range}(E)$ and $\mathrm{card}(F_\alpha) = \mu$.

(7) $E \neq \mathrm{id} \restriction F_\alpha$.

Proof. Since E is cofinal in F_β and $\mathrm{card}(F_\alpha) = \mathrm{card}(X) < \mathrm{card}(F_\alpha)^+ < \beta$ (by (5)). qed(7)

(8) $\mathcal{F}_\alpha$ is not a base for $\mathcal{F}_\infty$.

Proof. Otherwise the $\mathcal{F}$-strength of E yields a map $\pi_E\colon \mathcal{F}_\infty \to \mathcal{F}_\infty$ extending E. Then π_E is not the identity on L and this implies the existence of $0^\sharp$. Contradiction qed(8)

So there is a minimal fine level $\mathcal{F}_\gamma$ such that $\mathcal{F}_\alpha$ is not a base for $\mathcal{F}_\gamma$. Take $\nu < \alpha$ and a finite $p \subseteq \{F_\tau \,|\, \tau < \gamma\}$ such that $\mathcal{F}(F_\nu \cup p)$ is isomorphic to some $\mathcal{F}_\zeta$ with $\zeta \geq \alpha$. $\mathcal{F}_\zeta = \mathcal{F}(F_\nu \cup q)$, where q is the collapse of p. By the minimality of η we must have $\zeta = \gamma$, and we may assume:

(9) $\mathcal{F}(F_\nu \cup p) = \mathcal{F}_\gamma$.

By this property, γ must be a successor ordinal; let $\gamma = \delta + 1$.

(10) $\mathcal{F}_\alpha$ is a base for $\mathcal{F}_\delta$.

By the $\mathcal{F}$-strength of E, let

$$\pi_E\colon \mathcal{F}_\delta \to \mathcal{F}_\eta$$

be the extension of $\mathcal{F}_\delta$ by E. The extension map is fine up to $\mathcal{F}_\eta$ (6.3) and by 3.8 there is a fine extension

$$\pi_E^+ \colon \mathcal{F}_{\delta+1} \to \mathcal{F}_{\eta+1},\ \pi_E^+ \supseteq \pi_E.$$

Then

$$\begin{aligned} X &\subseteq \mathrm{range}(E) \subseteq \mathrm{range}(\pi_E) \subseteq \mathrm{range}(\pi_E^+) \\ &= \pi_E^{+\prime\prime} F_{\delta+1} = \pi_E^{+\prime\prime}\big(\mathcal{F}(F_\nu \cup p)\big) \\ &= \mathcal{F}((\pi_E^{+\prime\prime} F_\nu) \cup (\pi_E^{+\prime\prime} p)), \quad \text{since } \pi_E^+ \text{ is fine} \\ &\subseteq \mathcal{F}(F_{\nu^*} \cup (\pi_E^{+\prime\prime} p)), \quad \text{where } F_{\nu^*} = \pi_E^+ F_\nu,\ \nu^* < \beta. \end{aligned}$$

Setting $Y = \mathcal{F}(F_{\nu^*} \cup (\pi_E^{+\prime\prime} p))$ we have $Y \in L$, and $\mathrm{card}^L(Y) \leq \mathrm{card}^L(F_{\nu^*}) + \aleph_0 < \beta$.
This contradicts (1) above. **Qed**

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